A graphical representation of Arrow's theorem

Kenryo Indo

Kanto Gakuen University

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Aim of this slide

- Arrow's Social Welfare Function (SWF)
    - a function from the set of profiles of individual orderings into the set of social orderings satisfying the set of conditions which will be explained later.
- general impossibility theorem (Arrow, 1963).
    - Kenneth J. Arrow has developed the mathematical model of social choice and proved that dictatorship is unavoidable under a set of seemingly moderate conditions (i.e., the general impossibility theorem).
    - In this slide I will provide a brief graphical proof under linear orderings for 2-agent and 3-alternative as Part I. It can be intuitively understood, however, without loss of rigor. And I will discuss on general case in Part II.

Social choice theory

- Social choice problem (eg., voting/auction/...)
  - Alternatives (ex., candidates/commodities/...)
  - Agents (ex., voters/bidders/...)
  - Agent's possible preferences (ex., complete, transitive orderings)
  - A 'profile' is a tuple of each agent's preference.
  - Social decision rule (ex,. Condorcet rule/SPA/...)

Conditions of Arrow's SWF

- (T) Preference of each individual, or the society as a whole, is modeled as a linear (or weak) ordering, i.e., transitive, complete, asymmetric (or reflexive) binary relations on alternatives.
- (U) Unrestricted domain. Any profile (i.e., a combination of orderings of all agents) are possible.
- (IIA), (P), (ND) => next slide

Conditions of Arrow's SWF(2)

- (T), (U) => preceding slide
- (IIA) Independence of irrelevant alternatives. SWF is binary decomposable for each pair of alternative.
- (P) Pareto condition. Unanimity enforces the social decision.
- (ND) No-dictator. There is no unique agent who's ordering always to be a social ordering.

Arrow's theorem

- Theorem (Arrow, 1951/1963)
  - Let a model of n-agent and m-alternative, m>=3. And assume conditions U and T.
  - Then the set of conditions P, IIA, and ND for the SWF are inconsistent.
- Corollary

  P and IIA implies dictatorship (D).
- Observation

  Two dictatorial rules satisfy all these conditions.

  So, D is equivalent to P and IIA assuming U and T.

Part I

2-person and 3-alternative case

Binary decomposition which naturally represents the IIA condition

(a, b)

1 \ 2	>	<
>		
<		

(b, c)

1 \ 2	>	<
>		
<		

(c, a)

1 \ 2	>	<
>		
<		

For each pair (x, y),

> :- x is preferred to y

< :- y is preferred to x

The weak Pareto condition (unanimity)

(a, b)

1 \ 2	〉	〈
〉	〉 (shaded)	
〈		〈 (shaded)

(b, c)

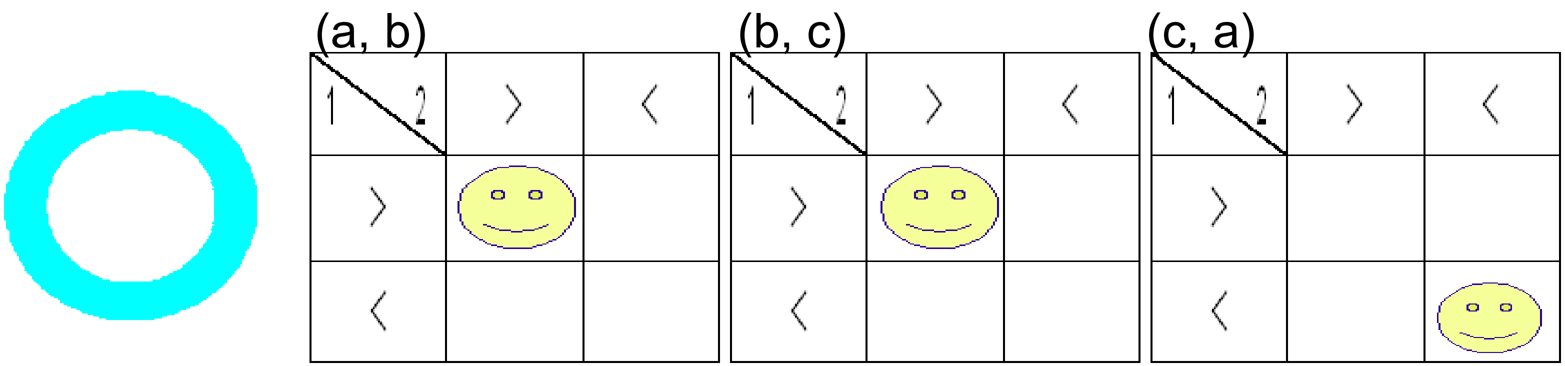
1 \ 2	〉	〈
〉	〉 (shaded)	
〈		〈 (shaded)

(c, a)

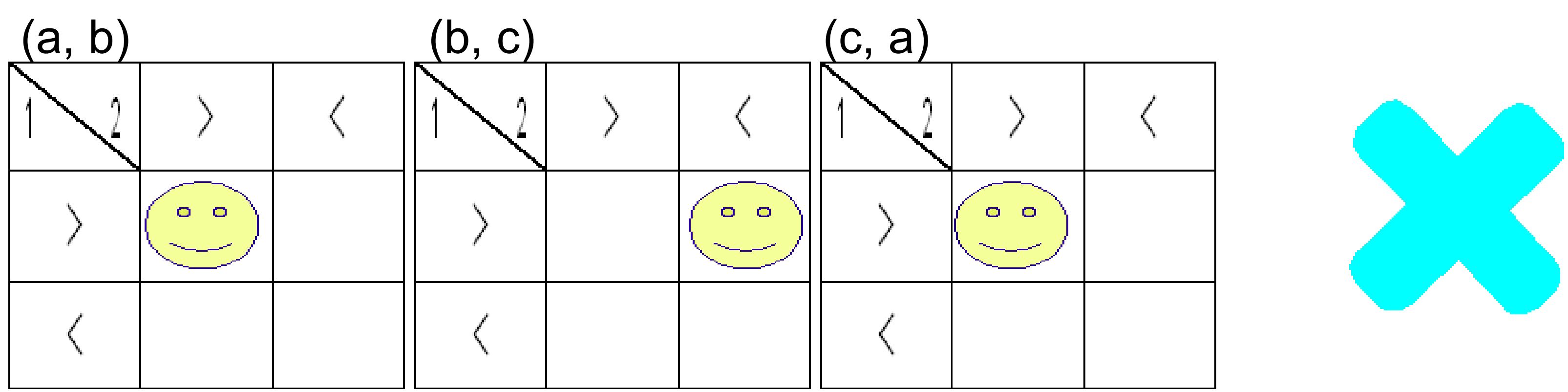
1 \ 2	〉	〈
〉	〉 (shaded)	
〈		〈 (shaded)

〉	:- x is preferred to y
〈	:- y is preferred to x
(shaded box)	: By Pareto condition

Profiles and the transitivity of individual orderings



Three smiles arranged by ones for each tables represent a possible profile, 1: (a>b,b>c,c<a) and 2:(a>b,b>c,c<a), a tuple of (transitive) orderings of two agents.



This is NOT a profile, because the ordering of row agent, 1: (a>b,b>c,c>a), is a cyclic relation, and so is intransitive.

Condition T prohibits each profile from being unilaterally directed

(a, b)

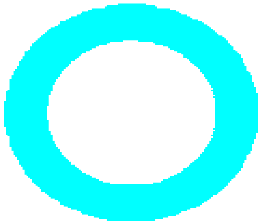
1＼2	>	<
>	>	
<		

(b, c)

1＼2	>	<
>		
<	>	

(c, a)

1＼2	>	<
>		<
<		



This can be seen as an SWF value assigned for a profile.

(a, b)

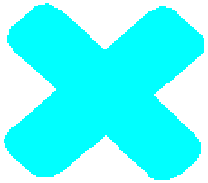
1＼2	>	<
>	>	
<		

(b, c)

1＼2	>	<
>		
<	>	

(c, a)

1＼2	>	<
>		>
<		



This can NOT be seen as a value of an SWF, because it consists a cyclic social orderings for the profile, and so is intransitive.

Condition T prohibits each profile from being unilaterally directed (2)

(a, b)

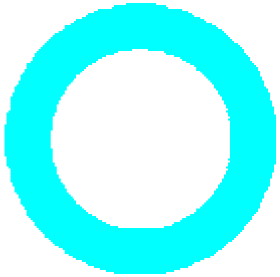
1 \ 2	>	<
>	>	
<		

(b, c)

1 \ 2	>	<
>	>	
<		

(c, a)

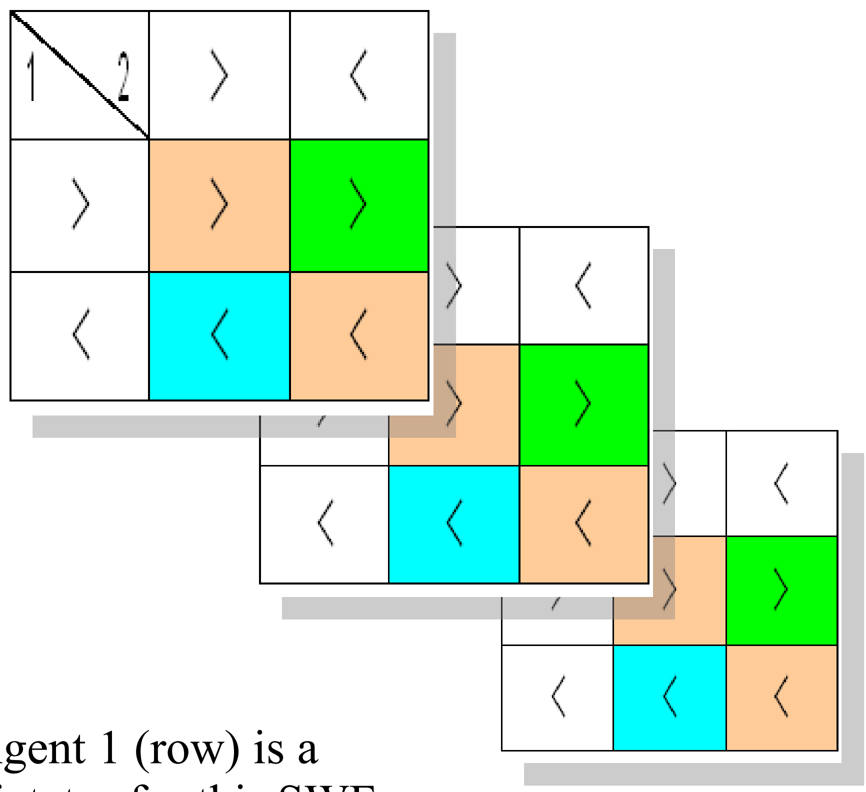
1 \ 2	>	<
>		
<	>	



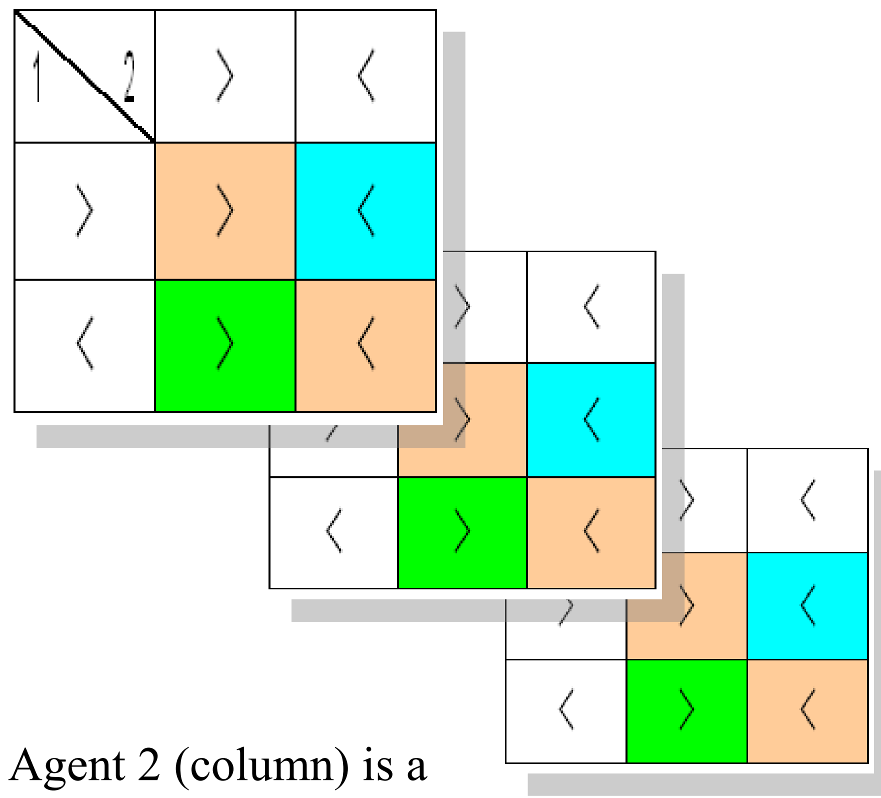
This does not violate Condition T because this is not a profile.

Two dictatorial rules

The dictatorial SWFs are clearly satisfies transitivity as well as other conditions of Arrow's theorem.



Agent 1 (row) is a dictator for this SWF.

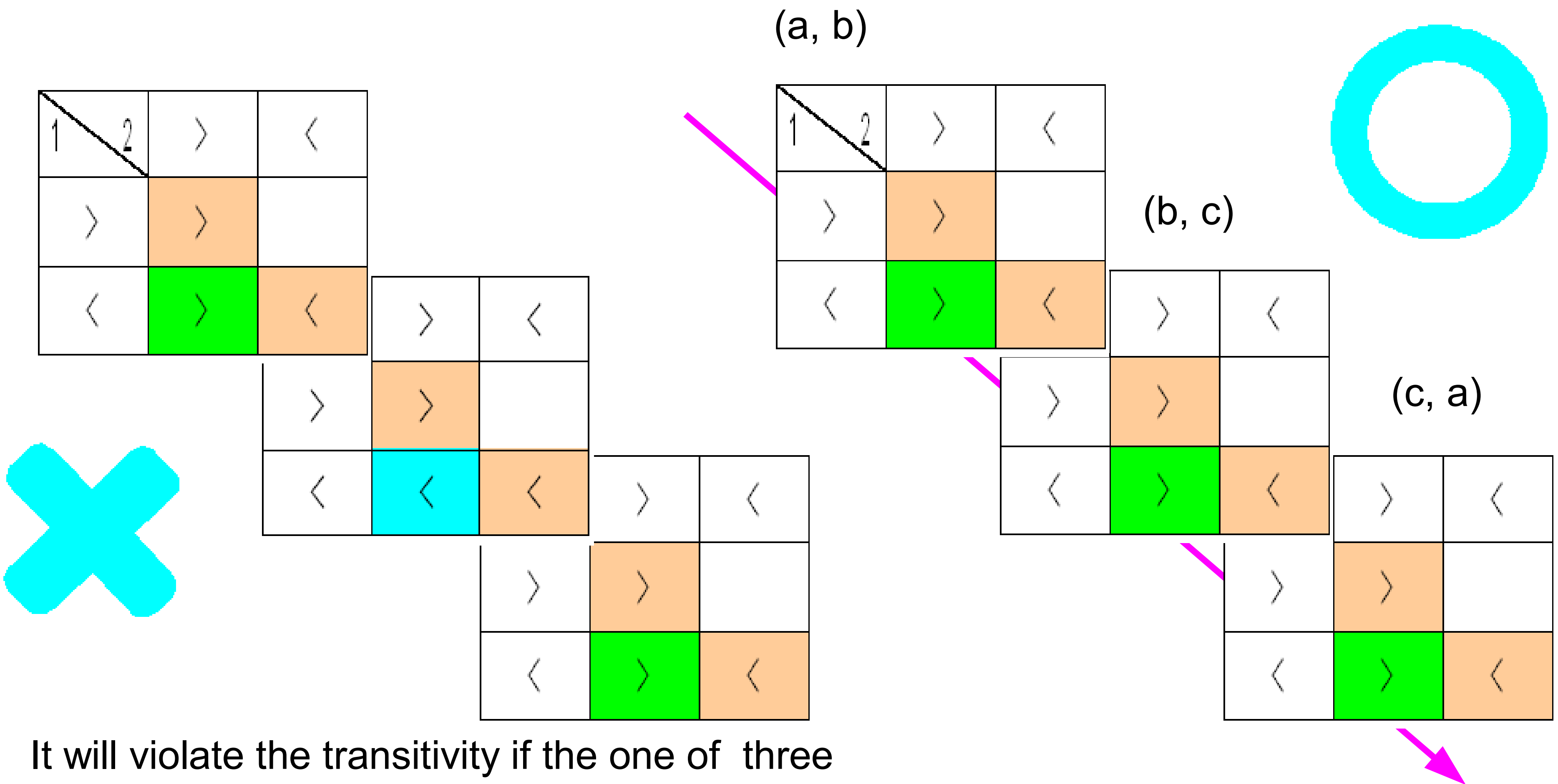


Agent 2 (column) is a dictator for this SWF.

Conditions of SWF restated graphically

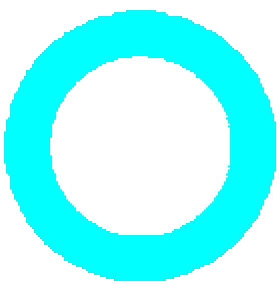
- (T & U) Individual ordering can not be selected within a single row (or a column) for each table. For each profile, which is a combination of such individual orderings, SWF should assign non-unilateral directions for each profile.
- (IIA) Profiles and SWF are represented by the three tables which are slices of the SWF with respect to directed pairs.
- (P) Diagonal elements of each table has a value which is same as the row and the column.
- (ND) There is a table which is not a simple duplications either of a row or of a column.

Condition T requires all tables to have a same single direction pushing through each non-diagonal cell (lemma 1)



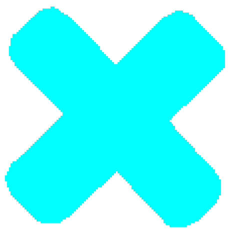
It will violate the transitivity if the one of three tables replaced with the above one.

Condition T implies that different non-diagonal elements should not be unilateral for each table (lemma 2)



1 \ 2	>	<
>	>	>
<	<	<

1 \ 2	>	<
>	>	<
<	<	<



You can not burn the candle at the both ends. It can be proved that it violates the transitivity!

Proof of the theorem

- Dictatorial rules clearly satisfies the conditions of SWF and above two lemmas.
- Obviously, lemma 1 and lemma 2 together complete a proof of the dictatorial result (and so of the impossibility theorem).

Proof (lemma 1)

I insist that we can suppose the following pattern of the SWF without loss of the generality. Then, I will prove that it goes to violate the transitivity.

(a, b)

1 \ 2	>	<
>	>	
<	>	<

(b, c)

1 \ 2	>	<
>	>	
<	>	<

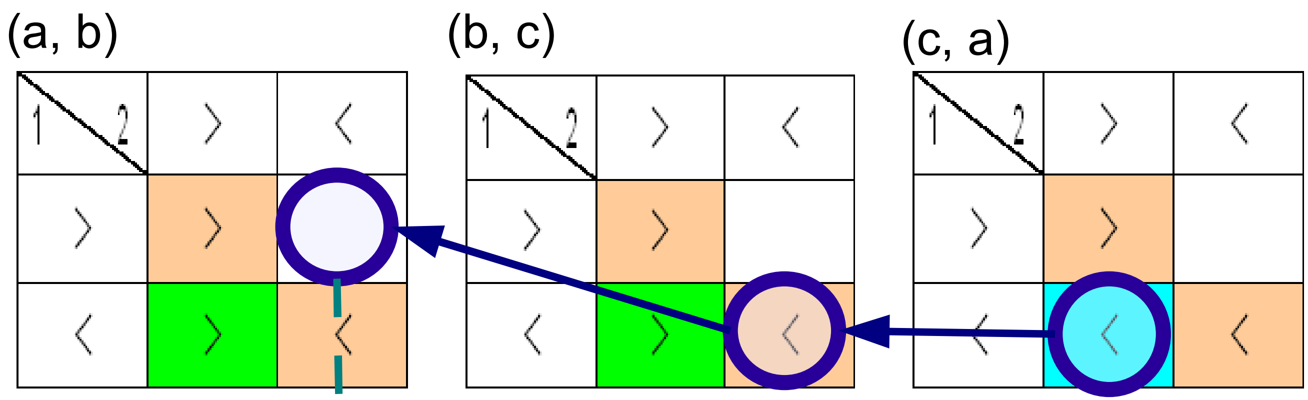
(c, a)

1 \ 2	>	<
>	>	
<	<	<

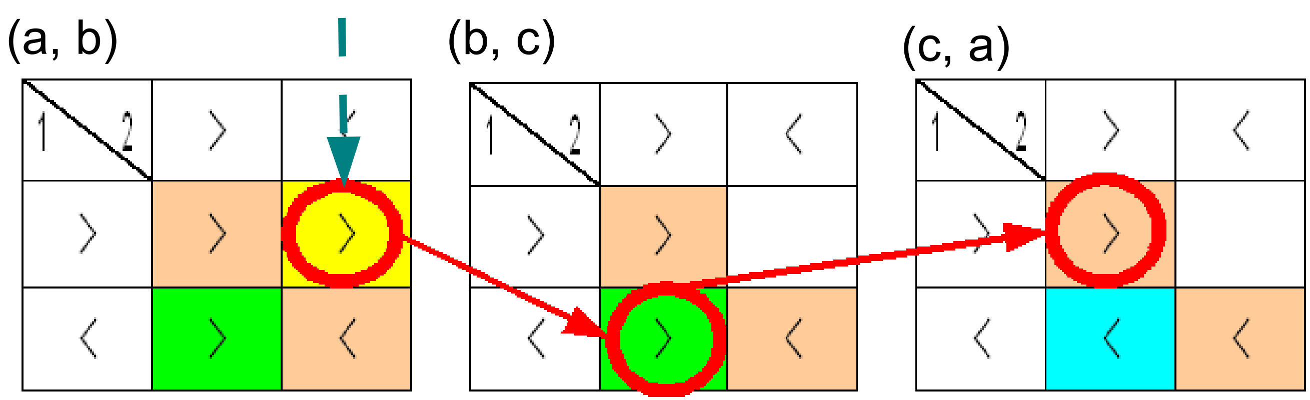
(next slide)

Proof (lemma 1) continued

Let us pick up a profile ((>,<,<), (<,<,>)). Then the value of the SWF must be a>b in order to satisfy Condition T.

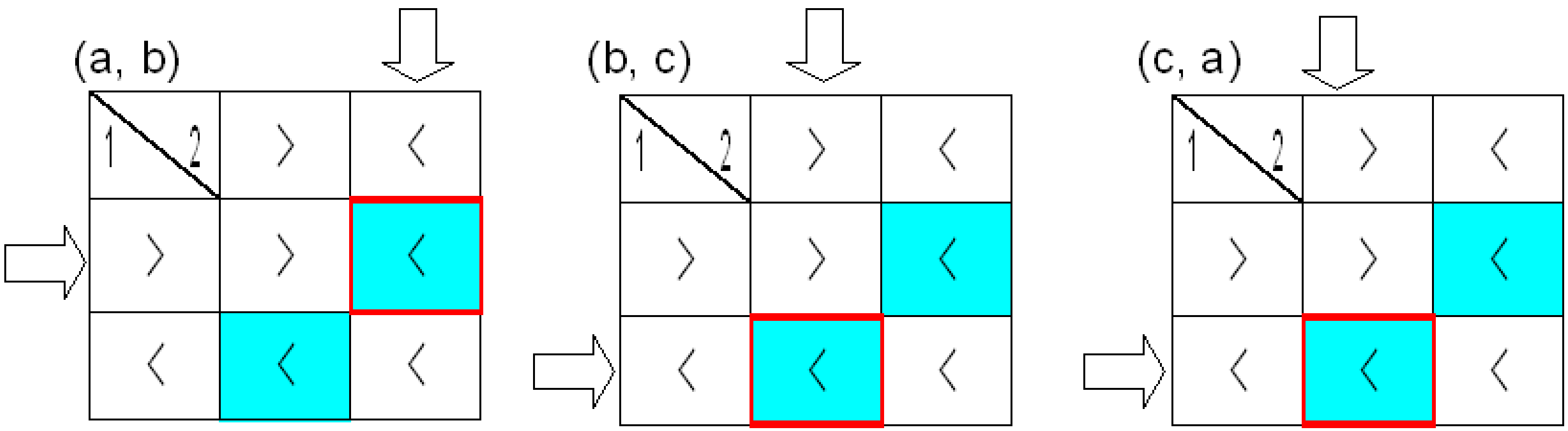


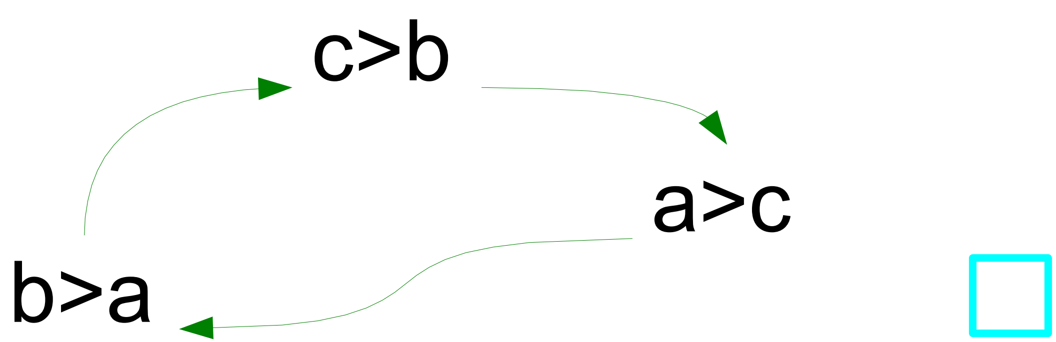
However it contradicts Condition T because a profile can be selected as shown in the following figure which shows an intransitive social ordering.



Proof (lemma 2)

Suppose a profile (a>c>b, b>a>c). By lemma 1, it suffices to consider an SWF like as the following pattern. This pattern results in a cyclic relation, so it can not be a social ordering.





Part II

General case

General case

- More than three alternatives
  - For every triple using same way in the preceding slides it can be proved similarly by renaming the symbols. And it will suffice to prove that for (1,2,3) and (2,3,4) there is a unique (local) dictator. It must be the case, for the two triples share a common pair (2, 3) and by Lemma 1 different dictator can not exist for any other pair.
- More than two individuals
  - As shown by Arrow (1963) or Sen (1995), the general case proof can be simplified by using decisive set and its decomposable nature (equivalently an ultra-filter).

decisive set

- A subgroup is said to be decisive if it enforces the social decision at every subgroup-unanimity profile. The maximal decisive set always exists as the unanimity (by Condition P). And this can sequentially be decomposed and shrink up to a singleton set where only a dictator included.
- You may find my graphical representation of SWF of some use also to understand how a special case of the impossibility can be extended to the general case proof naturally.

On proof of the theorem for n>=3

- I will only show a rough sketch of my proof for three individuals, or more, with some figures with new lemma which modified what of two-person case.
- Firstly, I introduce a modified representation of graphical SWF.

Another graphical SWF for n=2

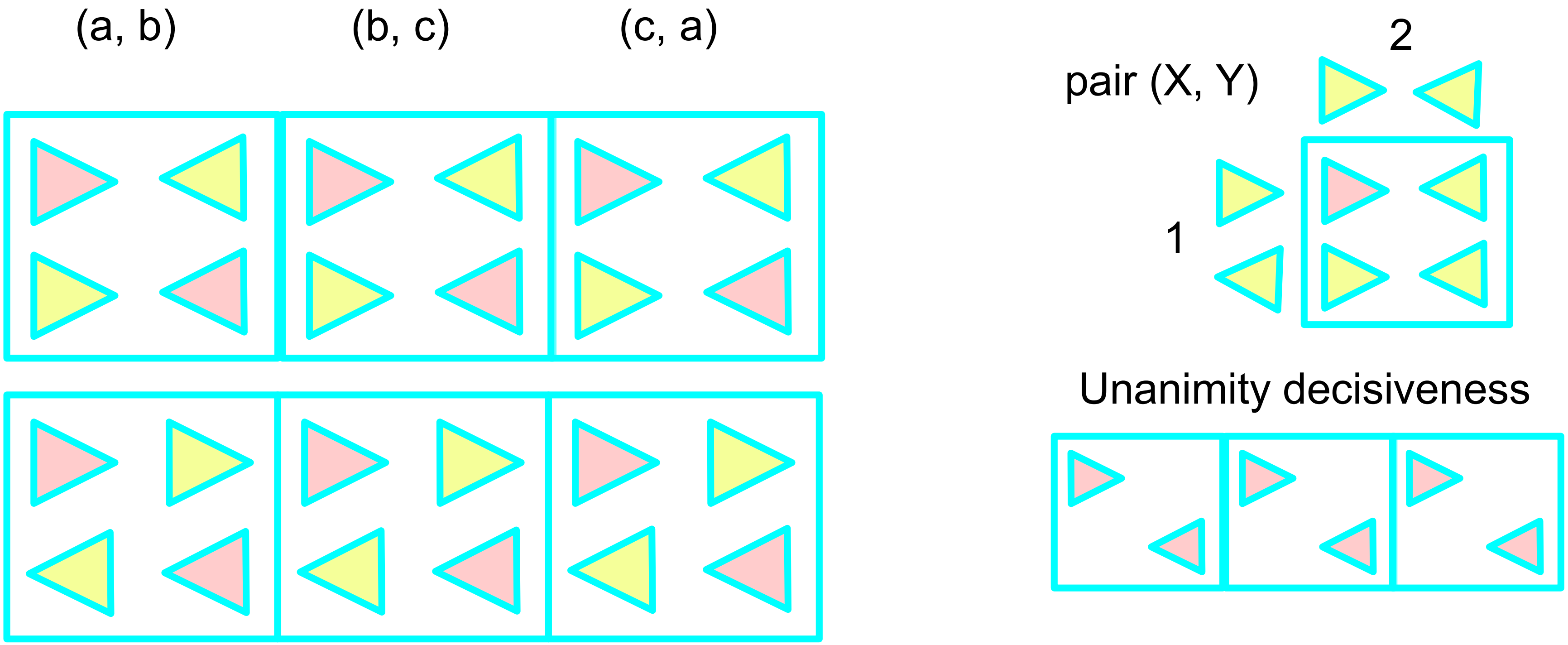


Fig. 2-1. I will use hereafter another graphical representation of SWF which is equivalent but somewhat concisely redesigned. Top and Bottom rows are all two-person SWFs two dictatorial rules, which can be seen as two possible decompositions of a unanimity decisiveness. Similar to the preceding tabular-styled representation, each row of three tiles with four small triangles represents an abbreviated preference profiles of 1 and 2. Each triangle with a direction in a tile to be interpreted as an social preference. Red triangles consist six unanimous profiles and should be Pareto condition obeying.

Three dictatorial SWFs for n=3

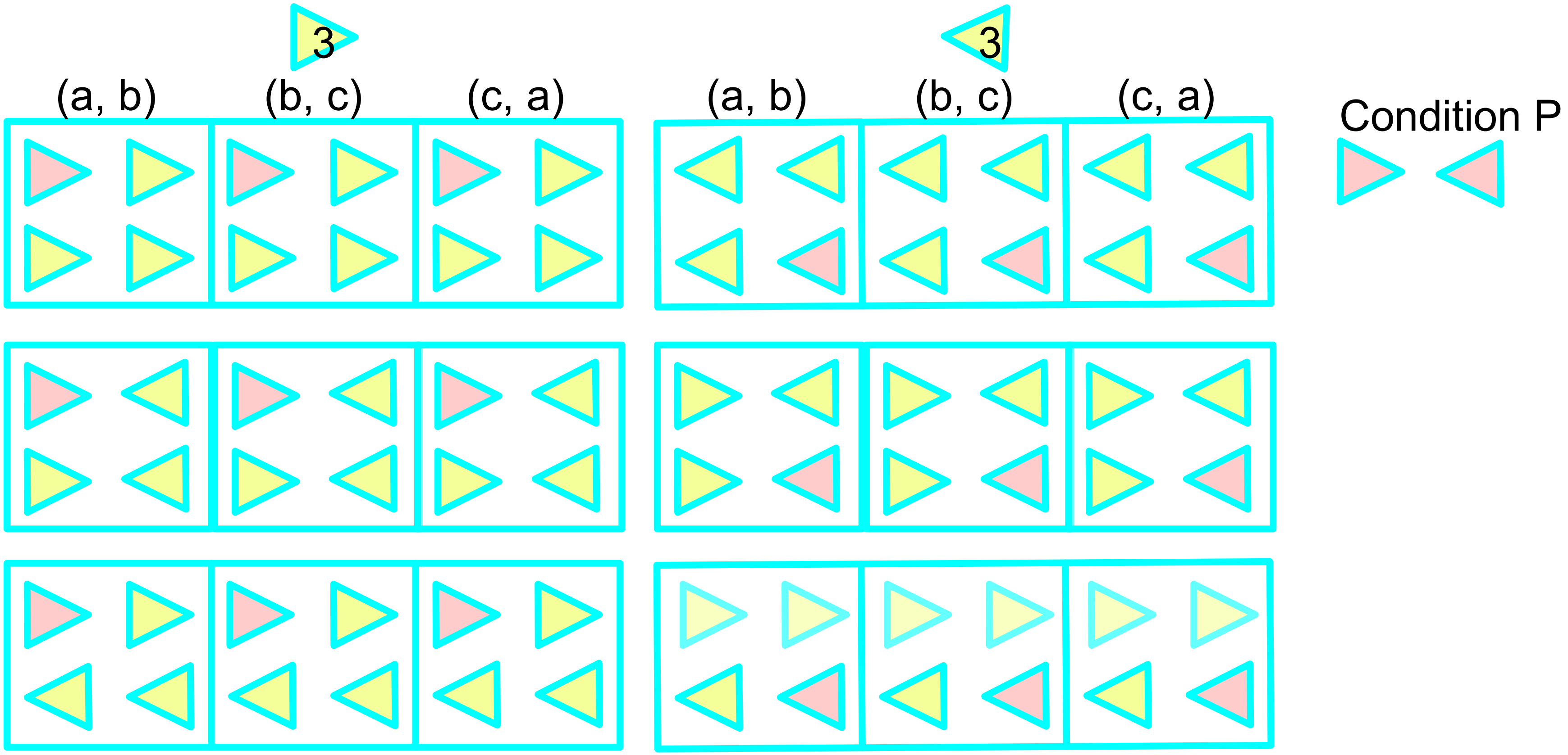


Fig. 2-2. Top, Middle, and Bottom rows are three-person dictatorial SWFs of individuals 3, 2, 1 respectively. These SWFs are decomposed into left and right parts according to the preference of individual 3. Each tile with four small triangles represents an abbreviated preference table for each pair as those used in two-person case, however, sub-profiles of 1 and 2 here. Each triangle with a direction in a tile to be interpreted as an social preference. Red triangles consists unanimity profiles.

(Lemma 1') Only three tiles

-- and their rotated images are used for every pair. And they are same single tile for each preference of individual 3.

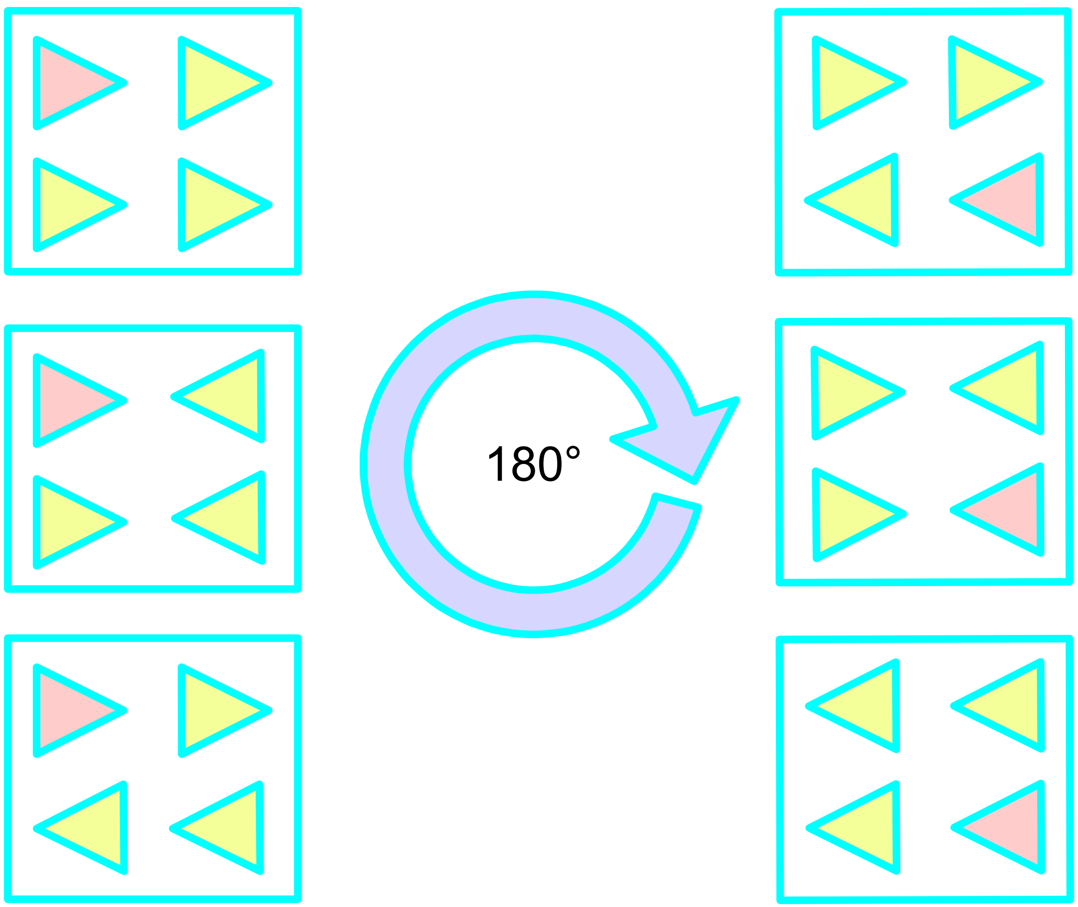


Fig. 2-3. Only three types of tiles are used in 3-person (and more person) SWF.

Changing focus on Lemma 1': Each row, or column, has to be sorted for each tile

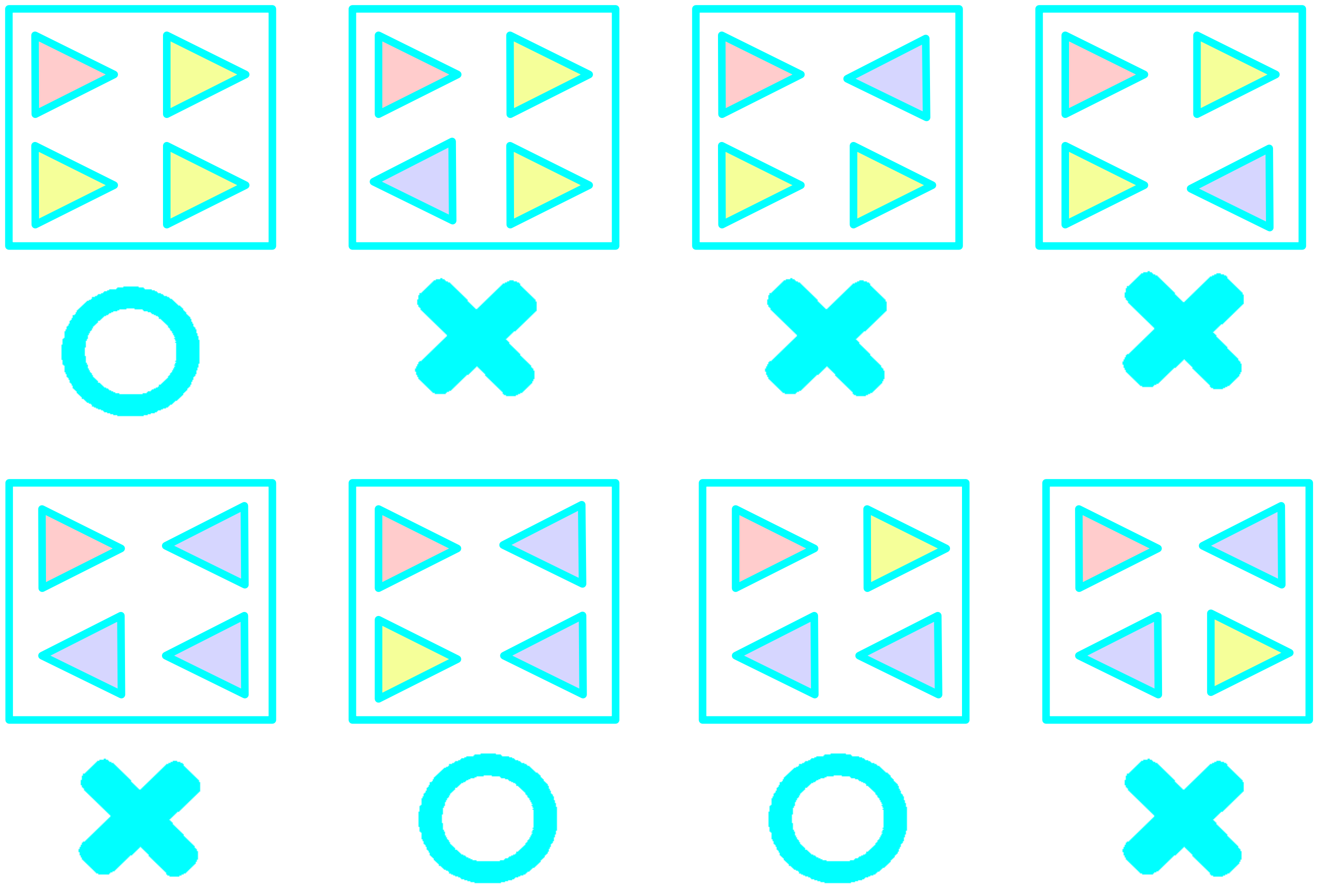


Fig. 2-4. Looking Lemma 1' in a different way: as a relaxation of Lemma 2. This lemma ca be seen as a relaxation of Lemma 1 and Lemma 2 of 2-person SWF.

How to decompose decisive set into dictatorship?

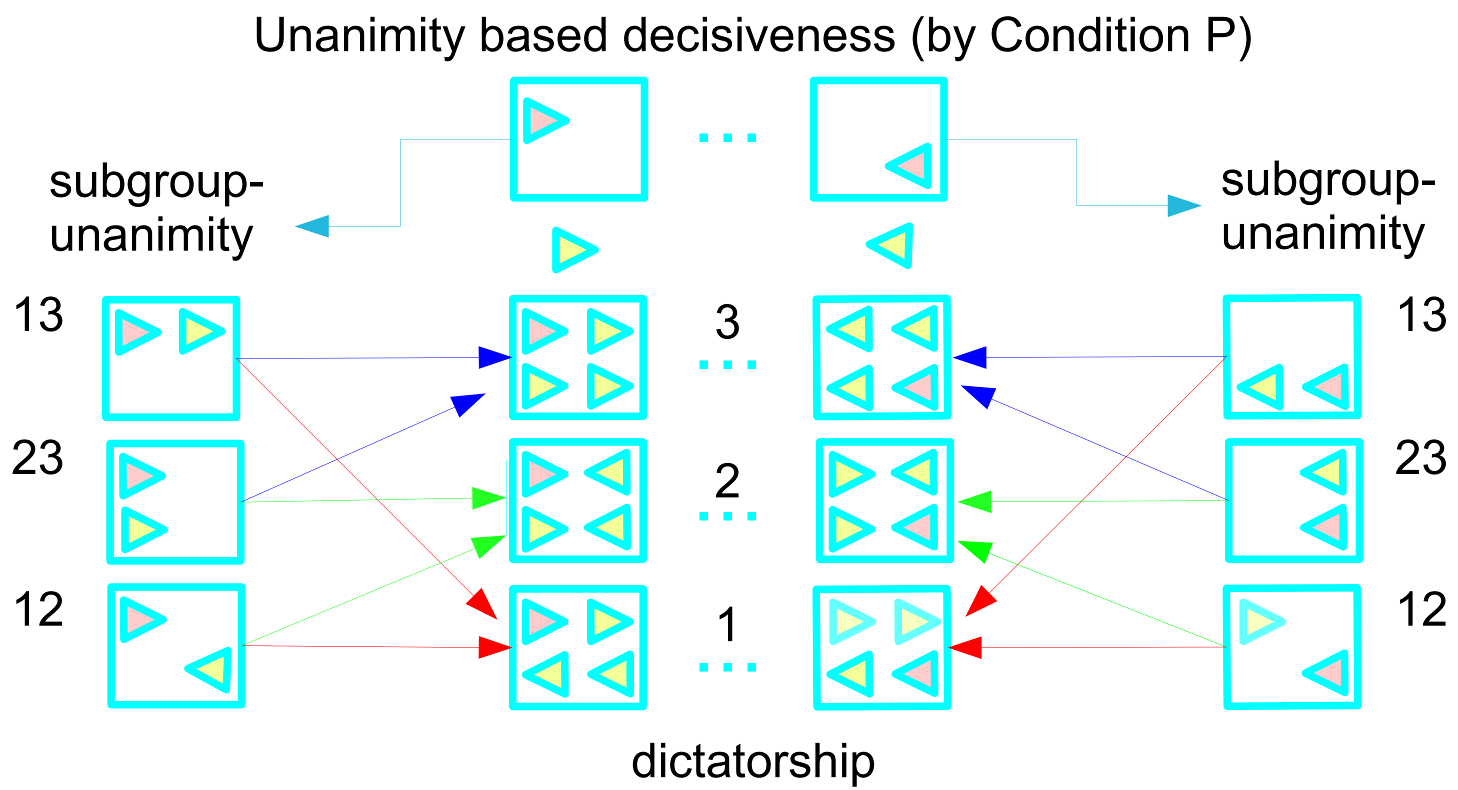


Fig. 2-5. Above figure explains intuitively how the decisiveness of unanimity has been decomposed into smaller (subgroup) decisive set for each pair (which is abbreviated in this figure). The intersection of any two 2-person subgroup unanimity decisiveness leads to a dictatorship by Lemma 1' and Lemma 2'.

dictatorial rules for n=4 and more

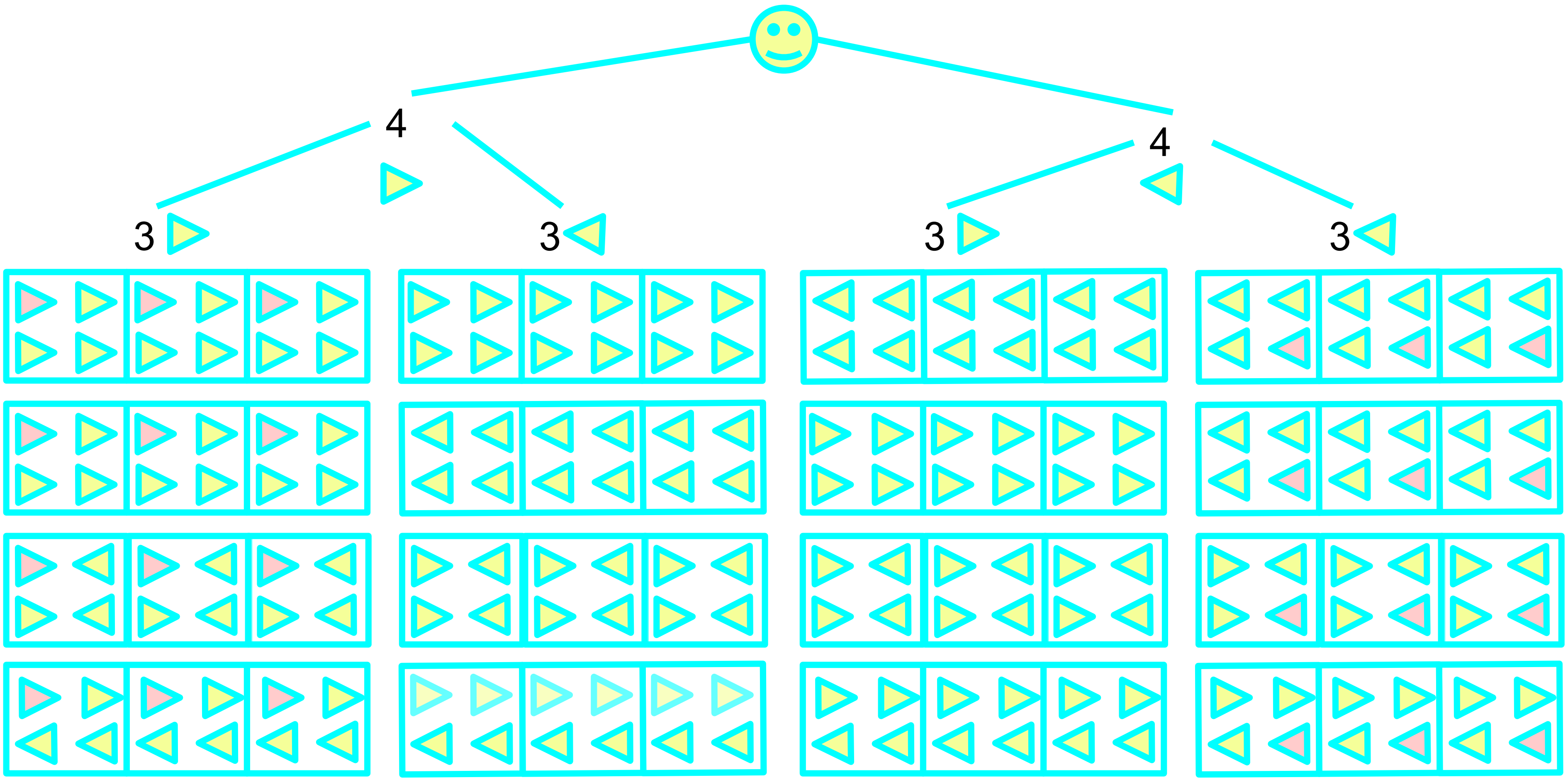


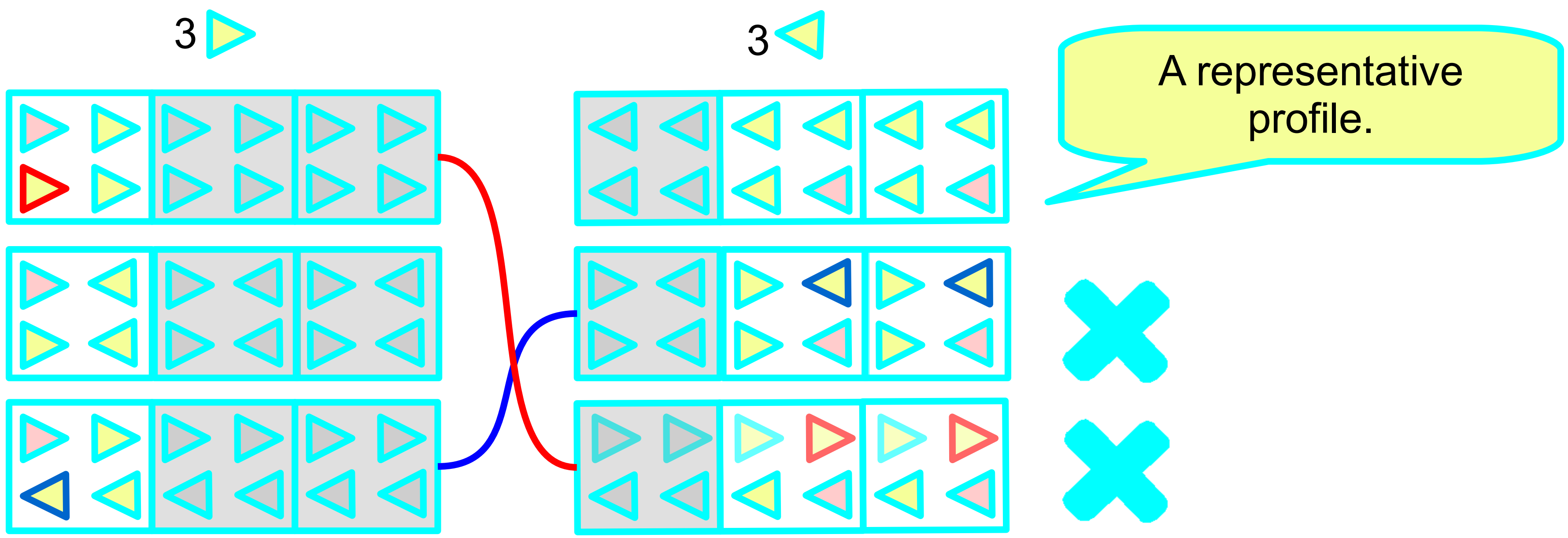
Fig. 2-6. Tree-and-table formed representation of SWFs for 4-person society. The top row a set of 12 tiles represents the fourth individual's dictatorial rule. Note that the graphical properties of dictatorial rule, therefore of Arrovean SWF, shown in the preceding slides stated as two modified lemma are invariant by adding a newcomer to the society of four or more individuals.

(Lemma 2') Cross-over of subsidiary dictatorial rules is not allowed

- Clearly, we can construct k-person SWF by adding a dummy agent and combine two subsidiary (k-1)-person SWF, k>=3, as well as a dictatorship of the newcomer as shown in Figure 2-6.
- That just goes to prove that there is no other case using mathematical induction.

Seemingly proved...

- Suppose there is a k-person SWF which is neither a (k-1)-dictatorial SWFs duplicated for each preference of k, a newcomer, or the k's dictatorship. The figure below is of k=3.



References

- K. J. Arrow (1951/1963). Social Choice and Individual Values, Yale University Press.
- A. Sen (1995). Rationality and social choice. American Economic Review 85(1):1-24.