

# A cube representation of social welfare function on weak ordering

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# Aim of this slide

- Kenneth J. Arrow has developed the mathematical model of social choice and proved that dictatorship is unavoidable under a set of seemingly moderate conditions (Arrow, 1951/1963). Often this theorem is called the general impossibility theorem.
- Arrow's Social Welfare Function (SWF) is a function from a set of product of orderings into a set of a (social) orderings satisfying the set of conditions which will be explained later.
- This slide extends the table and the cube-and-ball representations introduced by Indo (2007) and Indo (2008) respectively, by means of which the impossibility theorem for linear ordering 2-agent and 3-alternative case is proved, to the weak ordering case.

# Social choice theory

- Social choice problem (eg., voting/auction/...)
  - Alternatives (ex., candidates/commodities/...)
  - Agents (ex., voters/bidders/...)
  - Agent's possible preferences (ex., complete, transitive orderings)
  - A 'profile' is a tuple of each agent's preference.
  - Social decision rule (ex., Condorcet rule/SPA/...)

## Conditions of Arrow's SWF(1)

- (T) Preference of each individual, or the society as a whole, is modeled as a linear (or weak) ordering, i.e., transitive, complete, asymmetric (or reflexive) binary relations on alternatives.
- (U) Unrestricted domain. Any profile (i.e., a combination of orderings of all agents) are possible.

## Conditions of Arrow's SWF(2)

- (IIA) Independence of irrelevant alternatives. SWF is binary decomposable for each pair of alternative.
- (P) Pareto condition. Unanimity enforces the social decision.
- (ND) No-dictator. There is no unique agent whose ordering always to be a social ordering.

# Arrow's theorem

- Theorem (Arrow, 1951/1963)
  - Let a model of  $n$ -agent and  $m$ -alternative,  $m \geq 3$ . And assume conditions U and T.
  - Then the set of conditions P, IIA, and ND for the SWF are inconsistent.
- Corollary
  - P and IIA implies dictatorship (D).
- Observation
  - Two dictatorial rules satisfy all these conditions.
  - So, D is equivalent to P and IIA assuming U and T.

## Reducing into the set of orderings

- Note that what we can learn from the Arrow's theorem is that the set of possible profiles of all individual orderings is isomorphic to the set of all possible orderings under, SWF, the aggregation rule constrained by the above conditions.

# IIA condition and Pareto condition

- The IIA (Independence of Irrelevant Alternative) can be seen as the background constraint to draw the propagation of the transitivity relations, which is the most important condition in Arrow's impossibility theorem.
- The social orderings (SWF) have to be decomposed into the pairwise comparisons by IIA. See three 2x2 tables in the following figure. A profile of individual orderings is to select a unique cell from each table.

(a, b)

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(b, c)

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(c, a)

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> :- x is preferred to y

< :- y is preferred to x

 :- By Pareto condition

# Conditions of SWF restated graphically

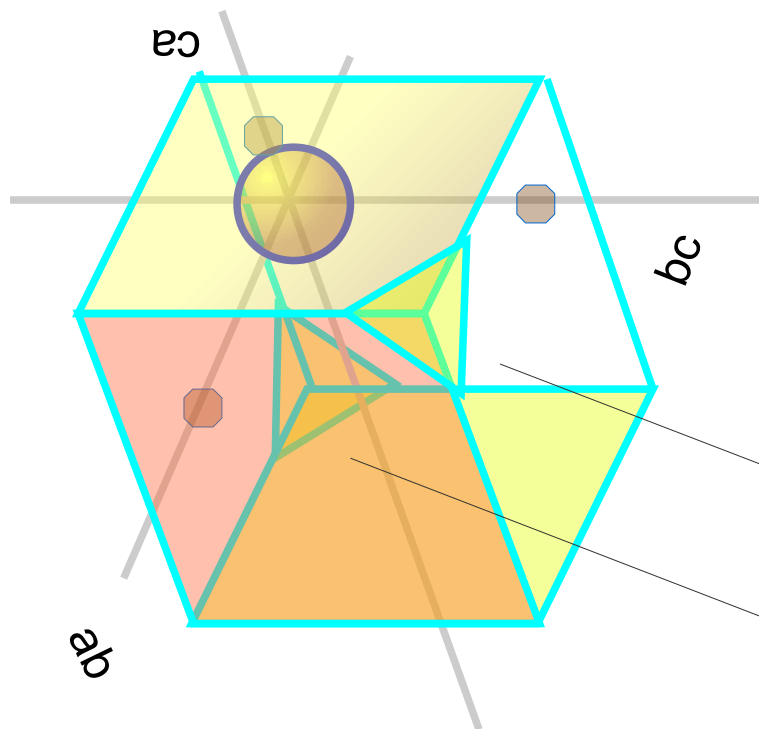
- (T & U) Individual ordering can not be selected within a single row (or a column) for each table. For each profile, which is a combination of such individual orderings, SWF should assign non-unilateral directions for each profile.
- (IIA) Profiles and SWF are represented by the three tables which are slices of the SWF with respect to directed pairs.
- (P) Diagonal elements of each table has a value which is same as the row and the column.
- (ND) There is a table which is not a simple duplications either of a row or of a column.

# Proof of the theorem

- Indo (2008) introduced the “A ball in a cube” representation (abbreviated to “cube”) for SWF and proved the Arrow's impossibility result.
- The reader should notice that for the original linear profiles are also profiles in weak orderings and the propagation process of transitivity is same. Therefore the SWF values are kept on this subset of profiles which consists of linear orderings.

# A cube representation of linear profile

- The following figure is one of the six cube in next slide represents an IIA-satisfied ordering profile. Three faces of cube you can see forefront corresponds to rotated tables used in the preceding slides.



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(c, a)

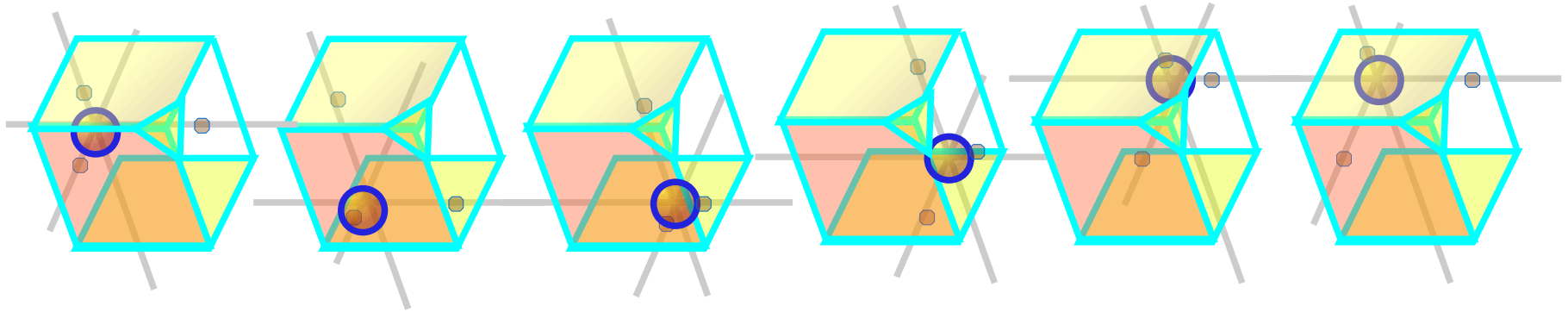
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You may not put a ball on special two corner positions of cyclic relation.

At this corner a ball would have corresponded to the All-“>”-position

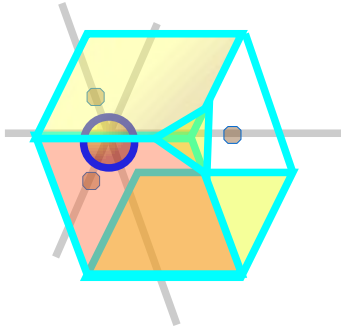
Another diagonal corner shows the All-“<”-position

# Ordering profiles of six balls



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# Transitivity propagation schema



- Transitivity (Condition T) inhibits all-same-vale, i.e., a cycle, for each profile. Therefore, if the value for a pair (a, b) and the binary profile ( $<$ ,  $>$ ) is “+”, which represents “ $>$ ”-social relation, in the figure left, then the value for a pair (c, a) at ( $>$ ,  $<$ ) can not be a “+” without a violation to Condition T.

(a, b)

$\backslash$	$\downarrow$	$>$	$<$
$>$			
$<$	<b>+</b>		

(b, c)

$\backslash$	$\downarrow$	$>$	$<$
$>$	<b>+</b>		
$<$	<b>P</b>		

(c, a)

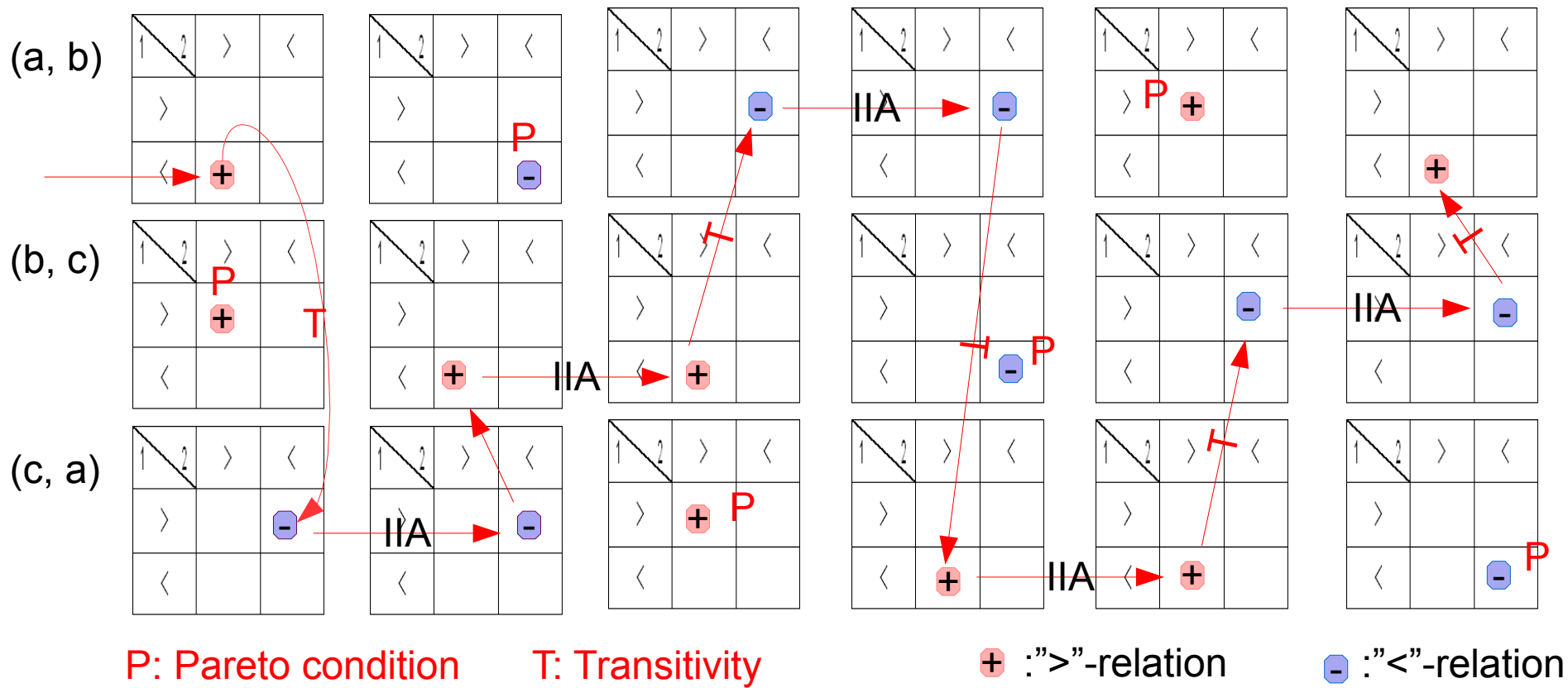
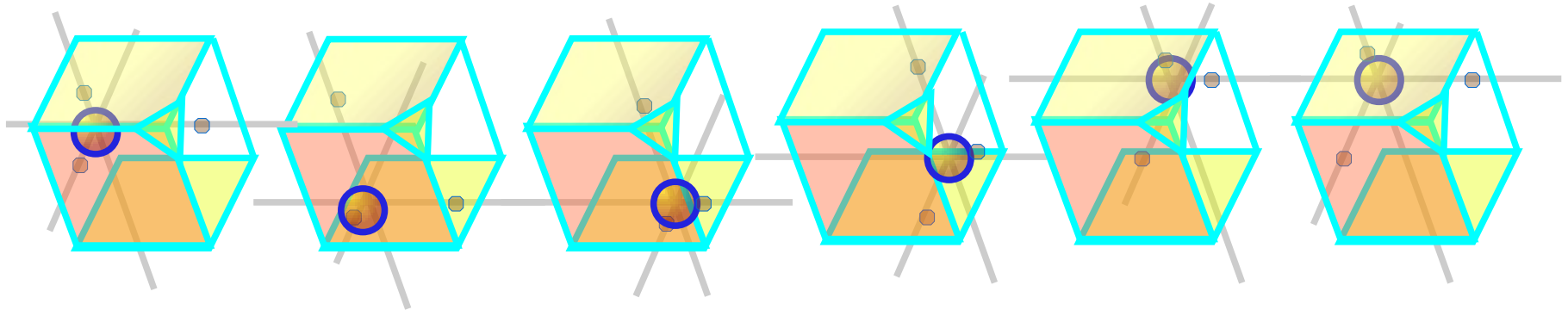
$\backslash$	$\downarrow$	$>$	$<$
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**+** :” $>$ ”-relation

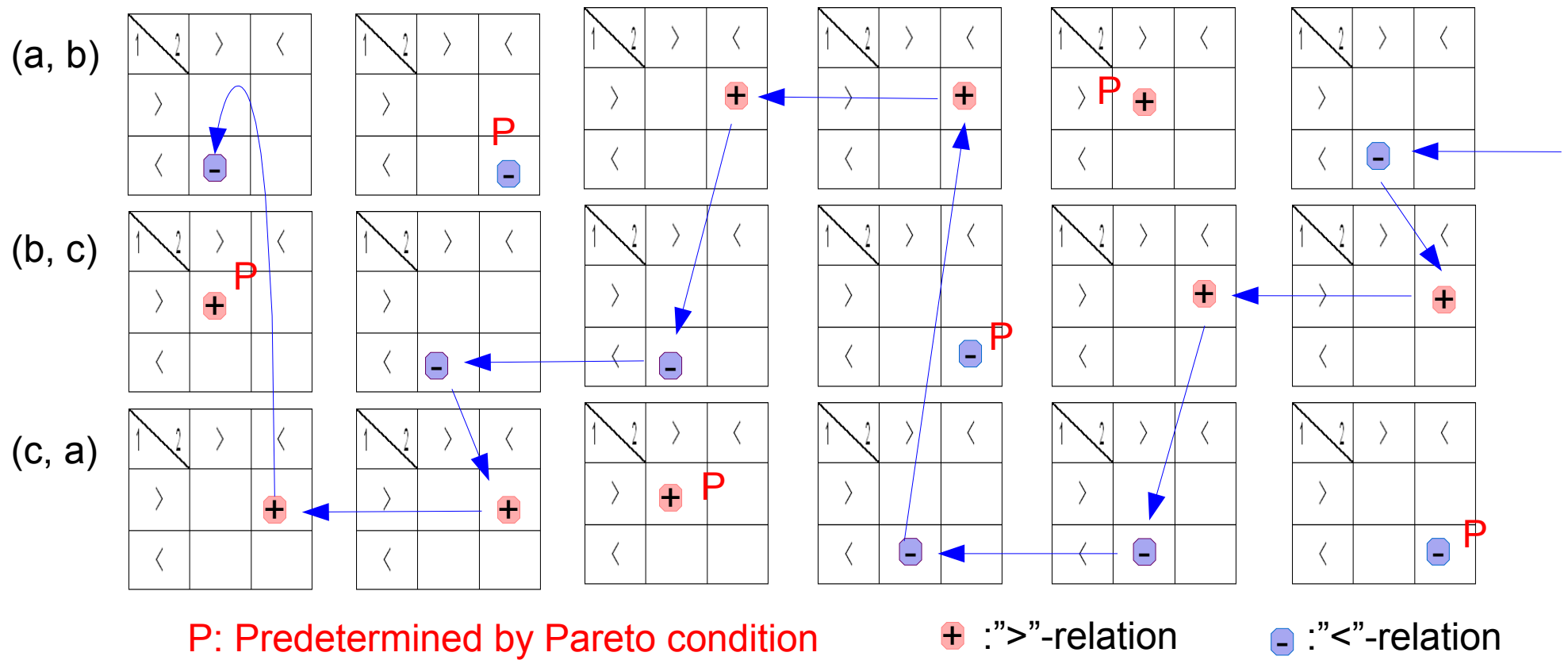
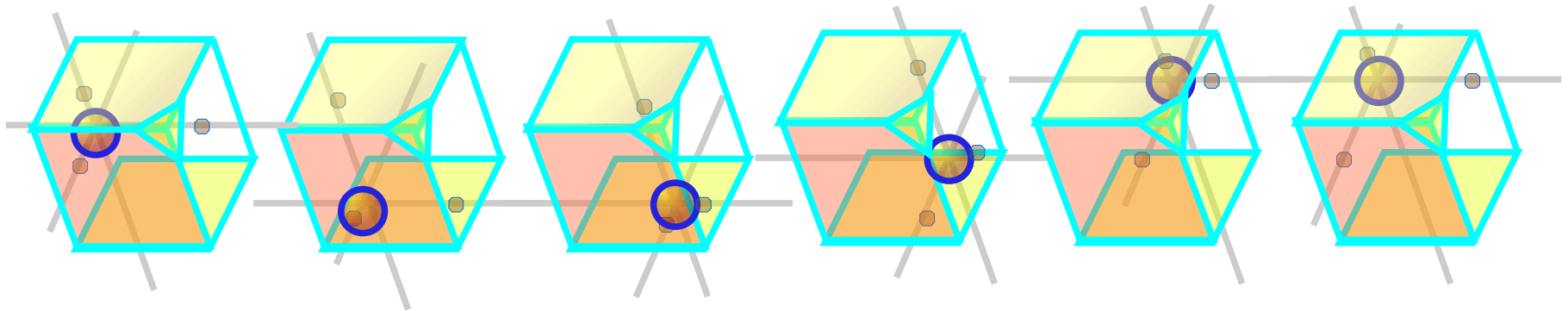
**-** :” $<$ ”-relation

**P**: Predetermined by Pareto condition

# Propagating transitive relations among cubes (1)



# Propagating transitive relations among cubes (2)

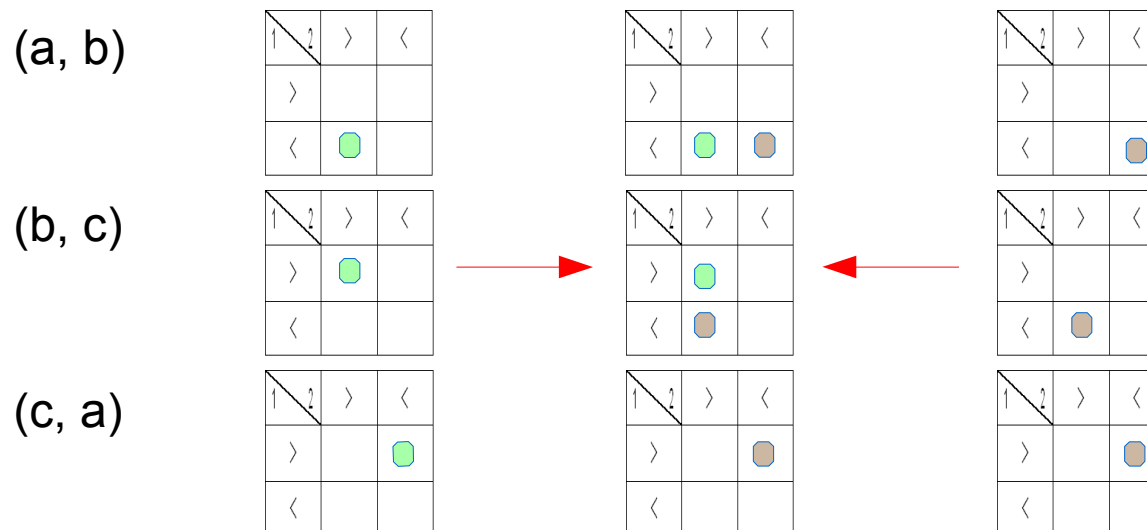
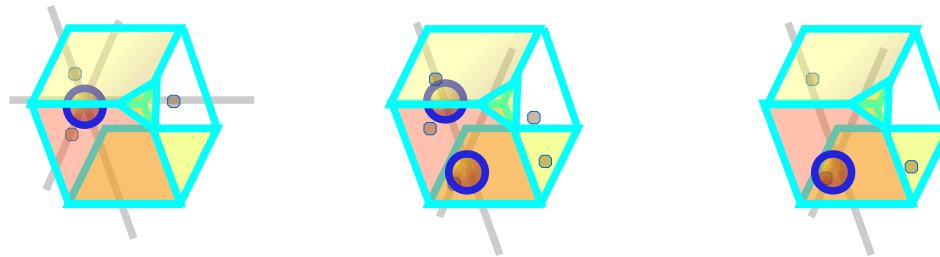


# Weak Orderings

- Arrow's original work is based on weak orderings, which permits indifferent relations in addition to the linear orderings. In my approach, a weak ordering profile can be represented by adding a possibility of the second ball, which should be adjacent to the first ball (or in all 6 positions).
- This operation means that a new cube comes from a composition with the adjacent cube (or the third next cube, in the sense of modulo). Seven new profiles which are not of linear ordering can be seen as transitional (and totally indifferent) states in this operation of composition.

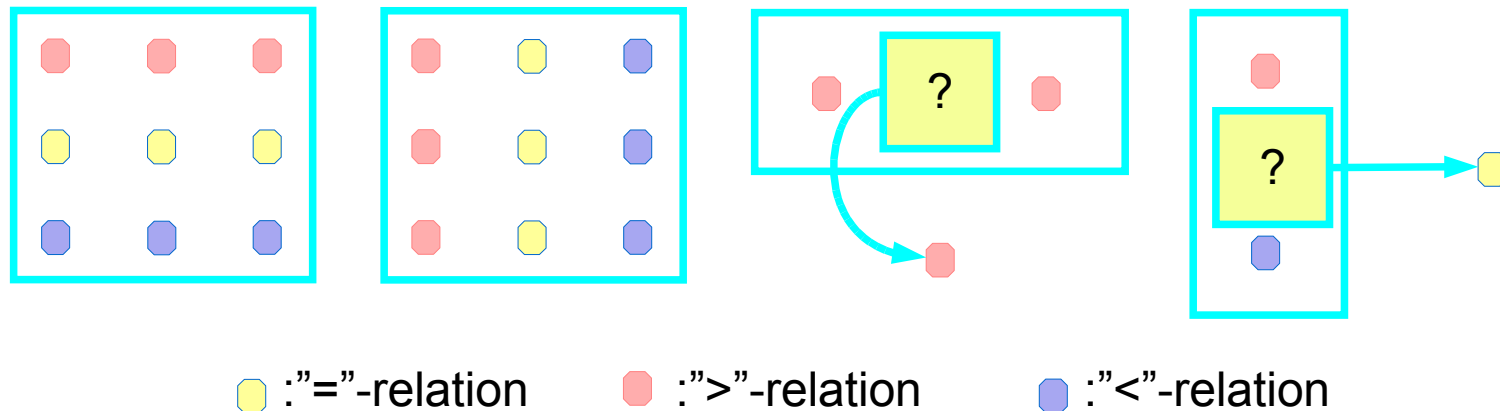
# Composition of cubes

- Each column (or row) in each face of a cube where two balls are appeared implies the middle position of up and down (or left and right).



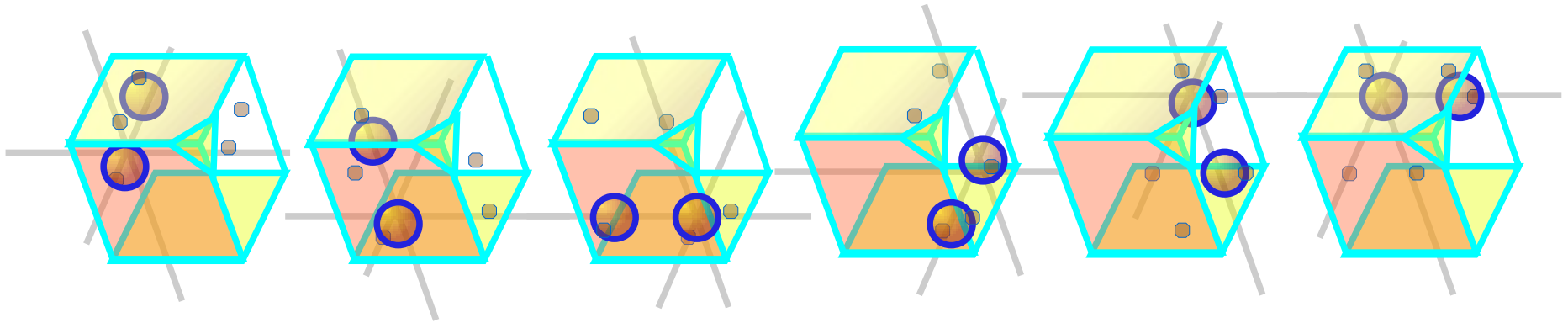
## On the table representation

- Alternatively, extending the tabular representation of SWF by Indo (2007), with middle positions in addition to the original four positions, up, down, right, left, it will suffice to consider patterns in the following figure.



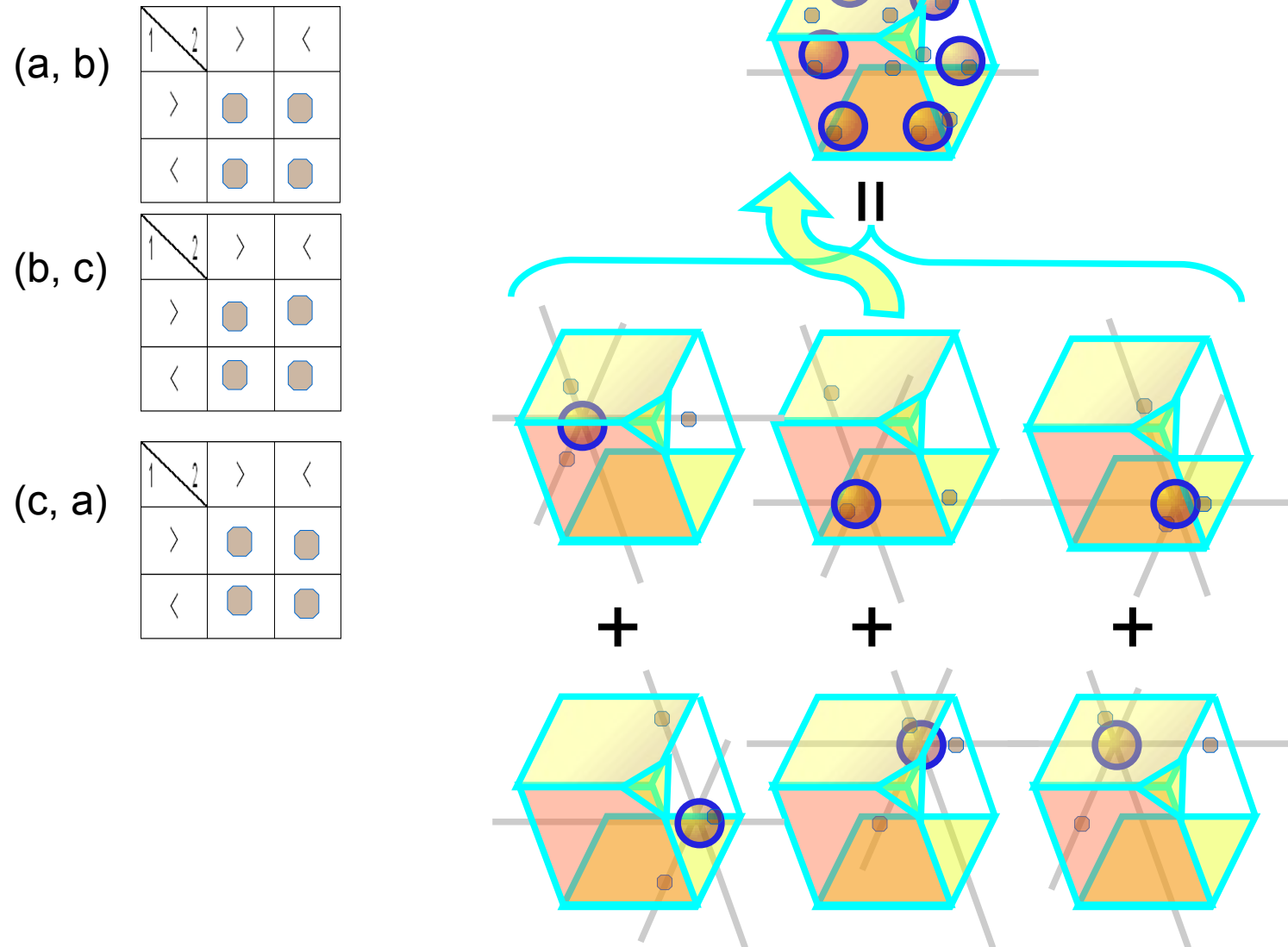
- Each middle position in a square, which intended to mean a sub-profile for a pair of alternatives, corresponds to a pair of balls at a cube in the previous figure.

# Additional weak ordering profiles



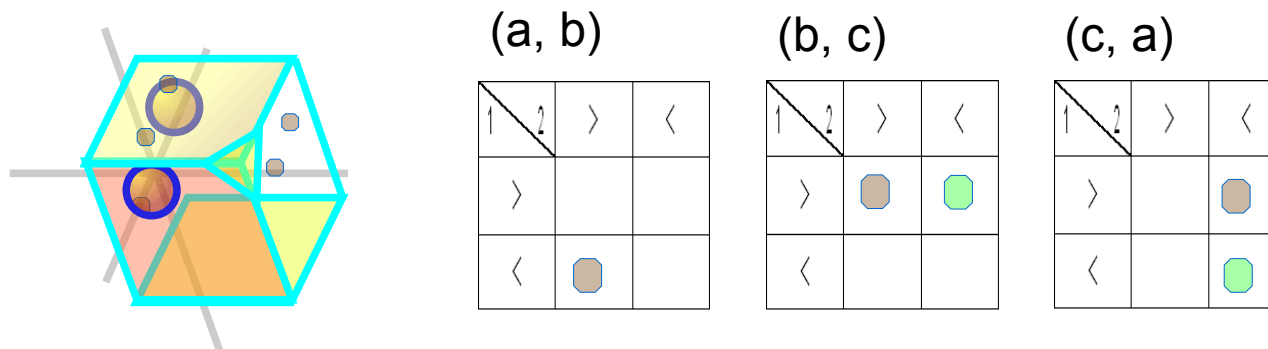
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# Totally indifferent profile



## Example of (in)transitive relations

- The following figure pick up the left most cube in the preceding slide, which represents a transitive profile, 1: ( $<$ ,  $>$ ,  $=$ ) and 2: ( $>$ ,  $=$ ,  $<$ ).



Note that the above profile is equivalent to 1: ( $<$ ,  $>$ ,  $>$ ) & ( $<$ ,  $>$ ,  $<$ ) and 2: ( $>$ ,  $>$ ,  $<$ ) & ( $>$ ,  $<$ ,  $<$ ) because of the IIA condition.

- In addition to the cycles, violations of the transitivity are such as, ( $>$ ,  $>$ ,  $=$ ), ( $<$ ,  $<$ ,  $=$ ), ( $=$ ,  $=$ ,  $>$ ), and so on. Similar restriction is imposed on the social orderings.

## Reconsideration on IIA

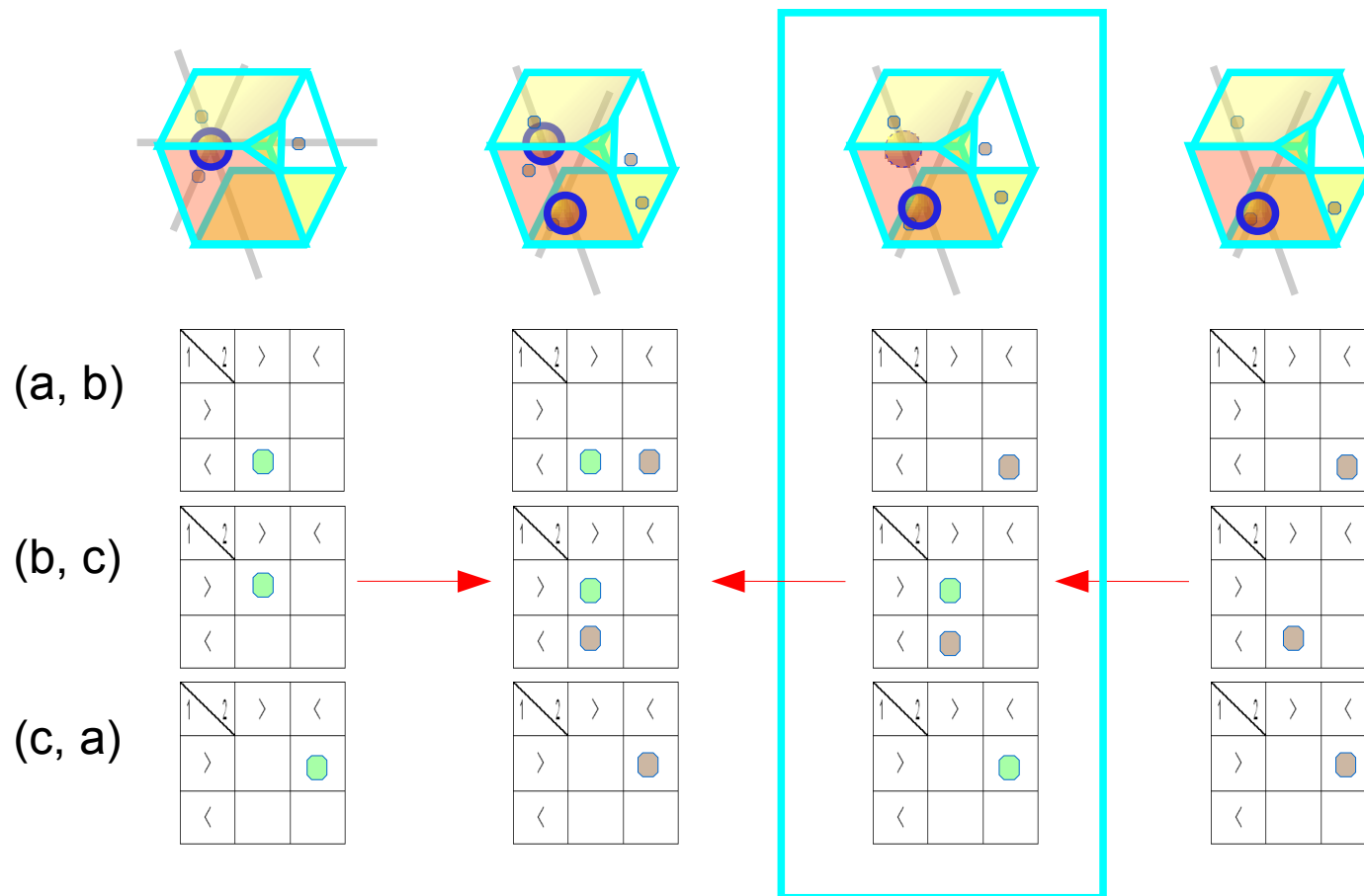
- In weak ordering it should be considered more carefully on the meaning of Condition IIA. It requires that it is independently decided not only to whether or not the ball is put on for each pair of alternatives but also whether or not for each room in a cube.
- Now before proving the theorem, the reader has to think a cube with more vivid image. A cube is comprised of eight rooms, i.e., a cube is decomposed into eight small cubes. Each room is independently represents the single direction of the relation “<” or “>” for each pair of alternatives and for each individual. Whether or not a ball in a single room is independent to any other ball and room.

## The proof

- Hereby we can reduce the  $13^2$  possible profiles into only 13 cubes (with their quarters). I will show that these are suffice to prove the Arrow's impossibility theorem.
- Note that, firstly, because the original linear profiles are profiles in weak ordering and the propagation process is same, the SWF values are same over every single ball face of a cube (i.e., sub-profiles regarding a pair of alternatives).
- Secondly, for each double-ball face, because of the IIA condition the propagation process of transitivity should assign a same SWF value as the linear components of each ball for each pair of alternatives.

# Quarter cube

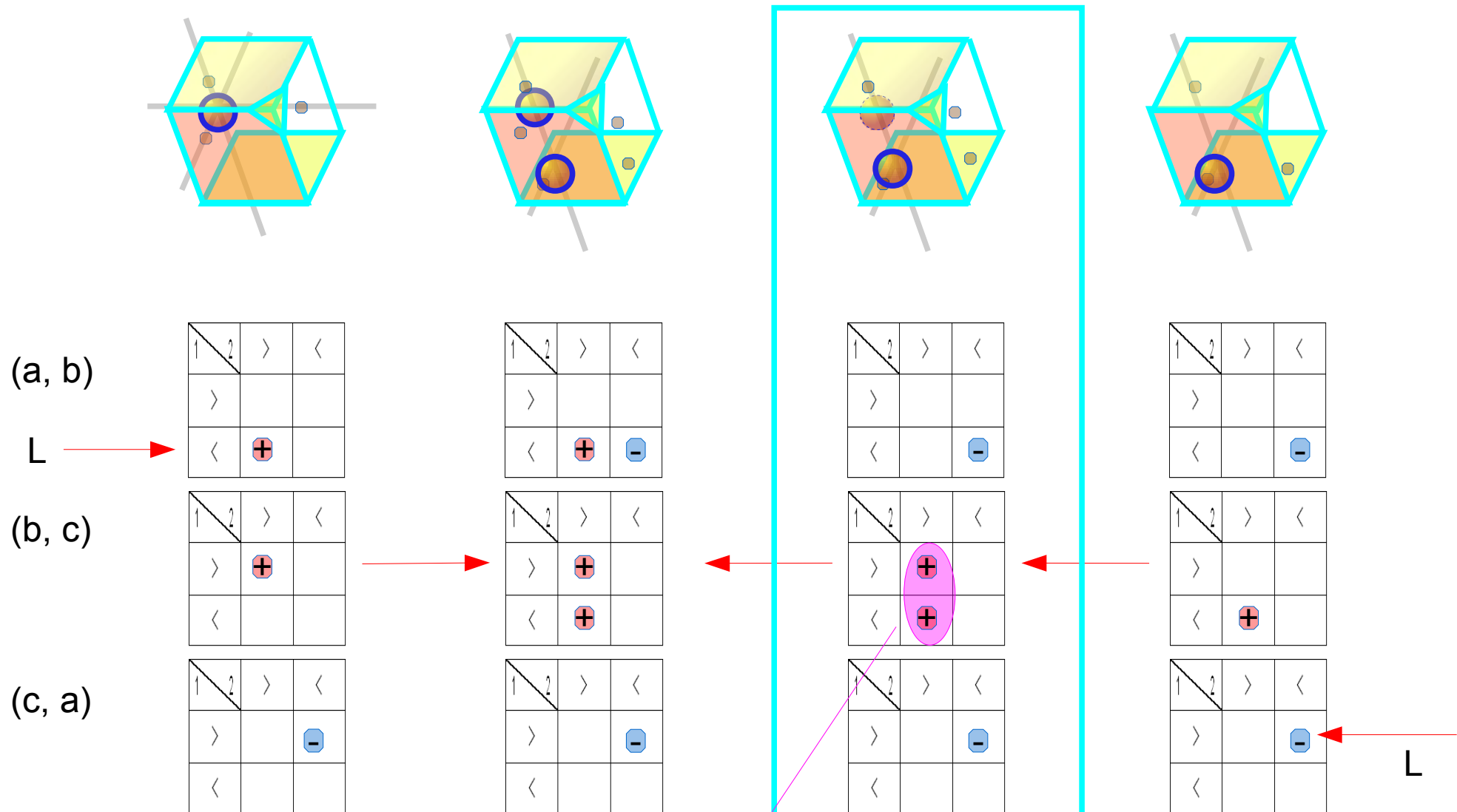
- A quarter cube is a being between a linear profile and its adjacent weak profile, which also represents a profile without violation of Condition T.



# Lemma 1

- For each middle position, which is either a row or a column of a double-ball cube, if both sides agree on and it is not indifference then a SWF must assign the same relation.
- Proof. By using quarter cubes and transitivity propagation schema as follows.

# Propagation process via quarter cube (1)



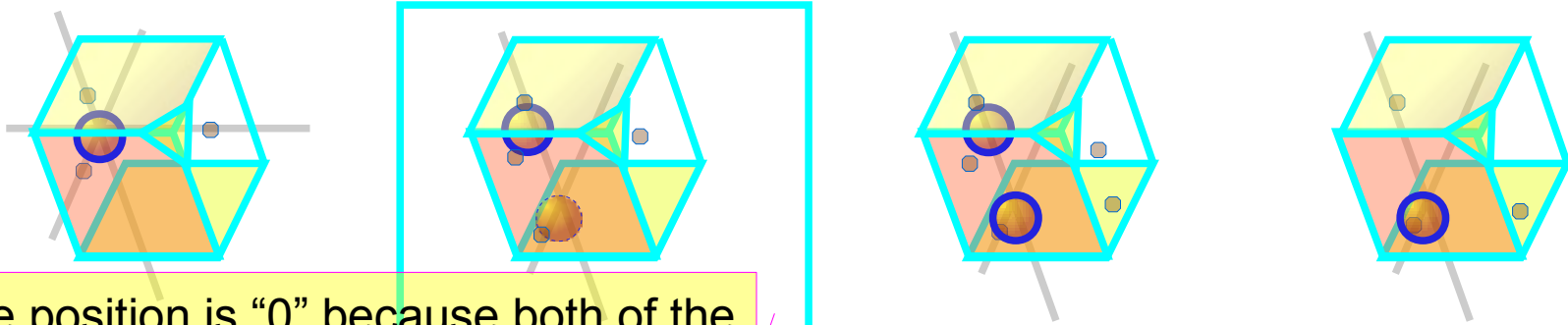
This middle position must be “+” for each component of individual 1's preference.

L: From the linear part of the SWF.

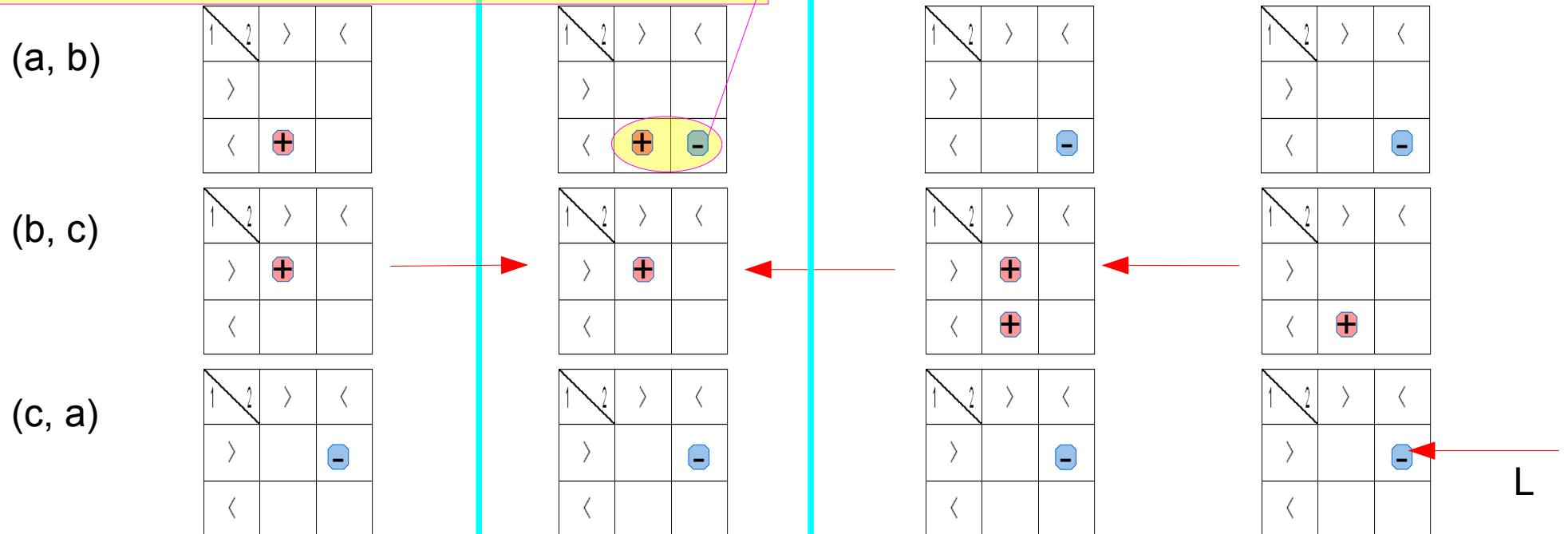
## Lemma 2

- For any middle position which is being between of different directions “<” and “>” should be a “=” relation.
- Proof. As follows.

# Propagation process via quarter cube (1)



This middle position is "0" because both of the two directions can be derived.

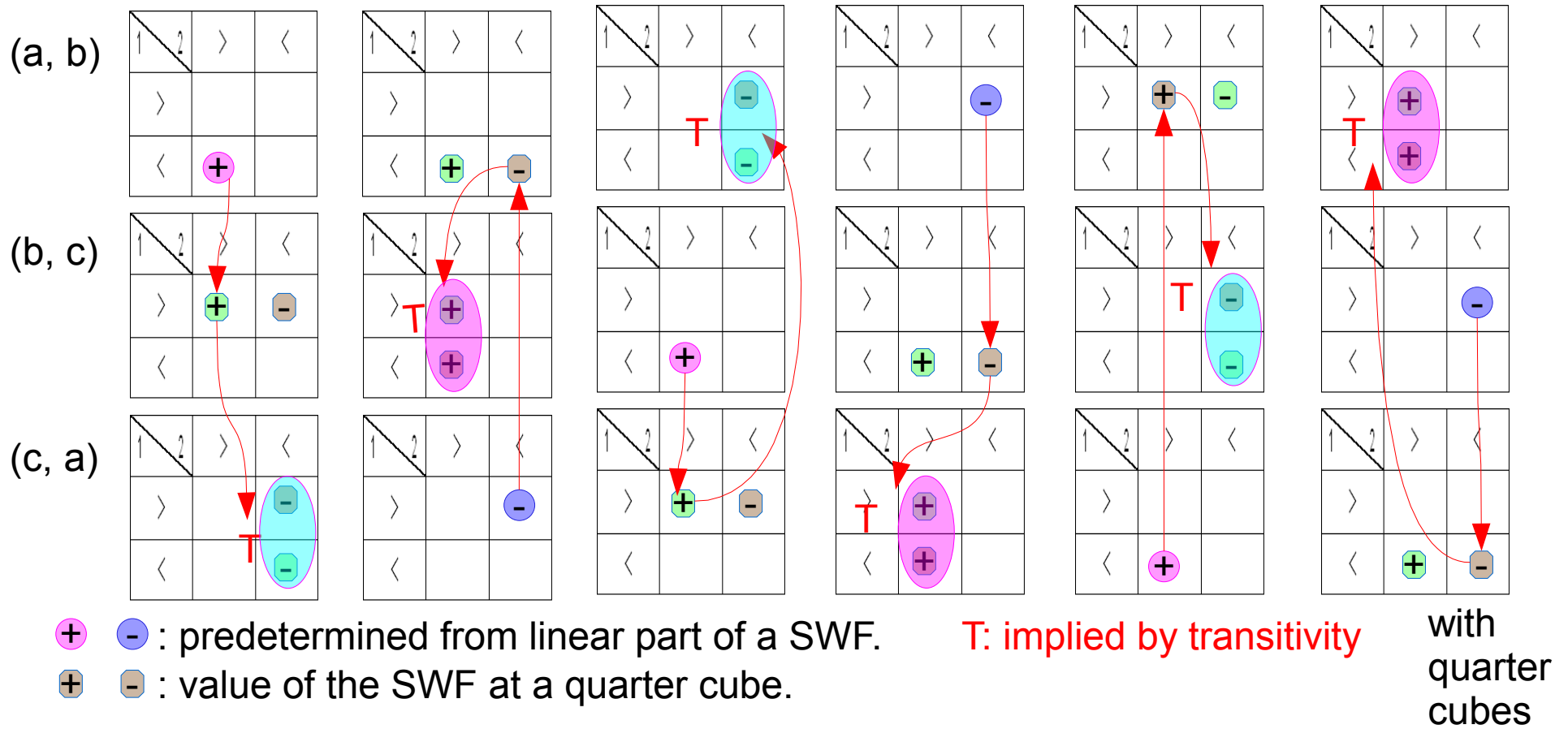
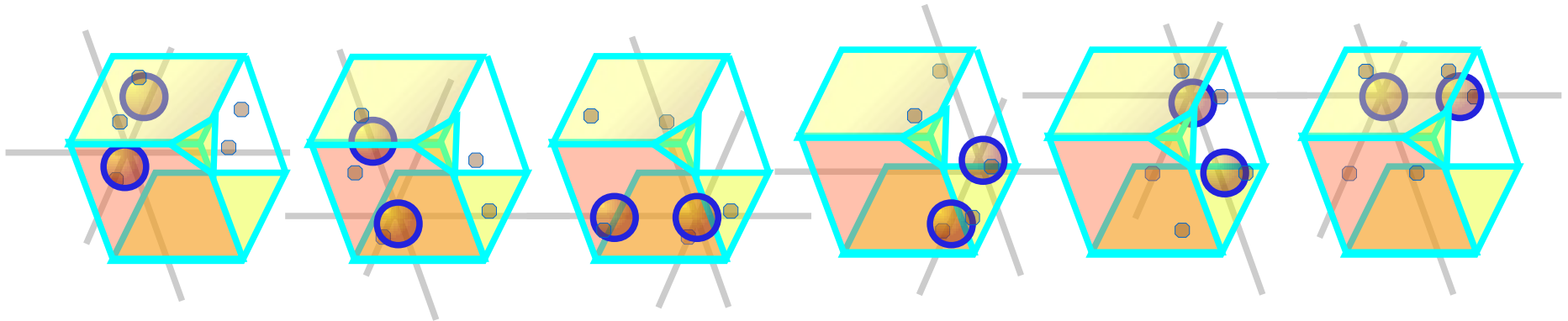


L: From the linear part of the SWF.

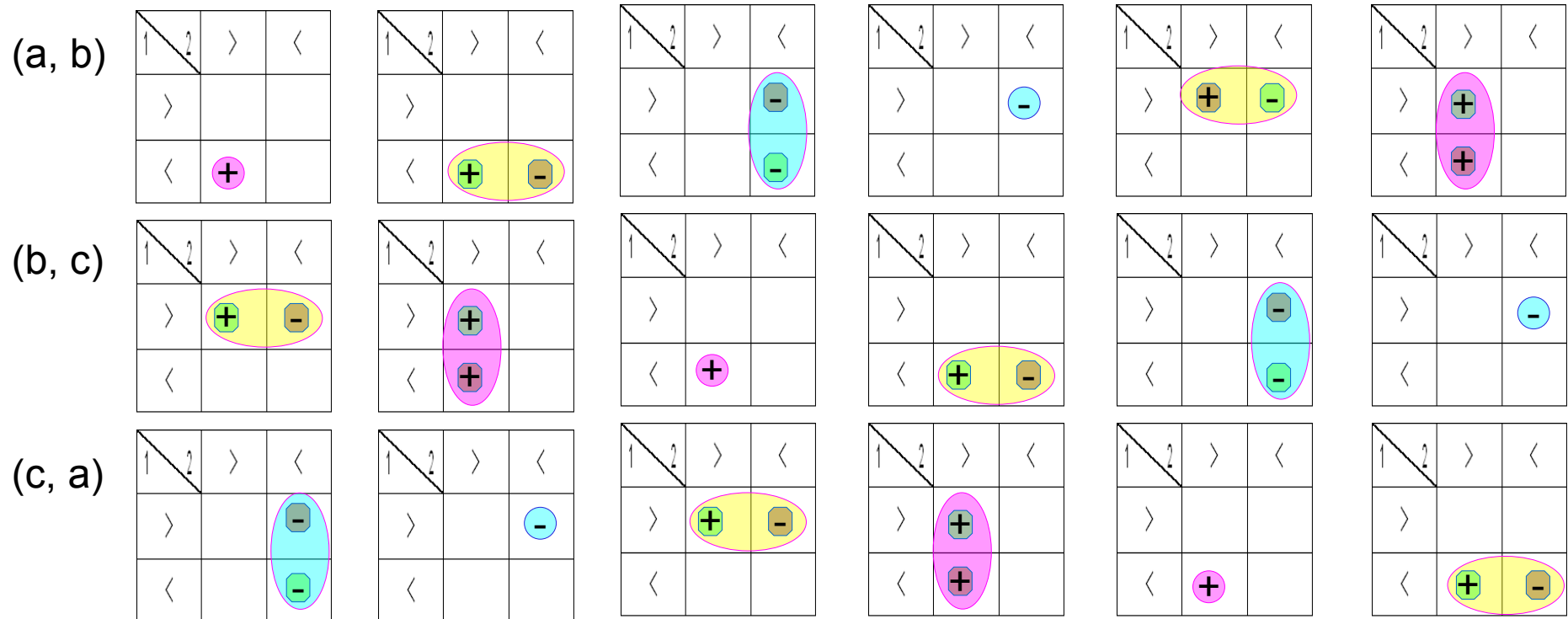
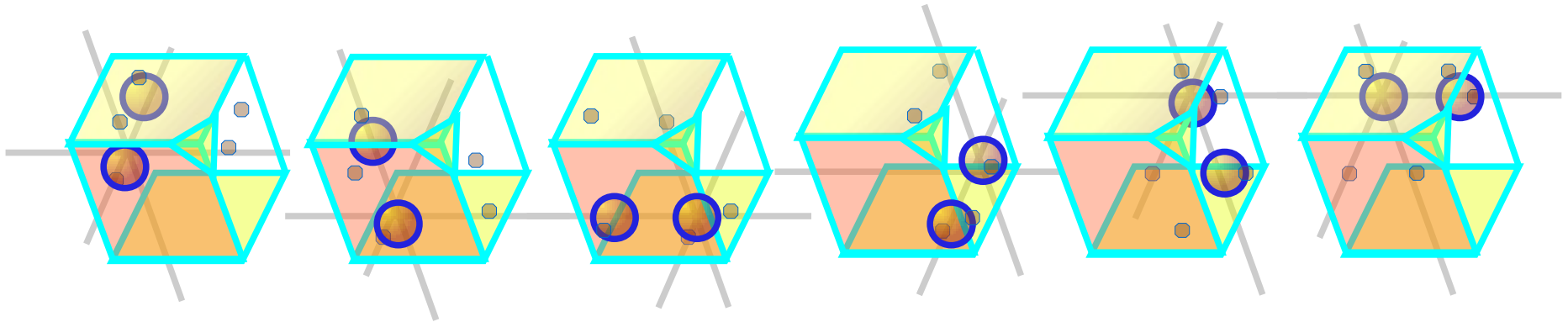
# Proof of the theorem

- Proof. By using the preceding two lemmas repeatedly as follows.

# Propagation process (1.1)



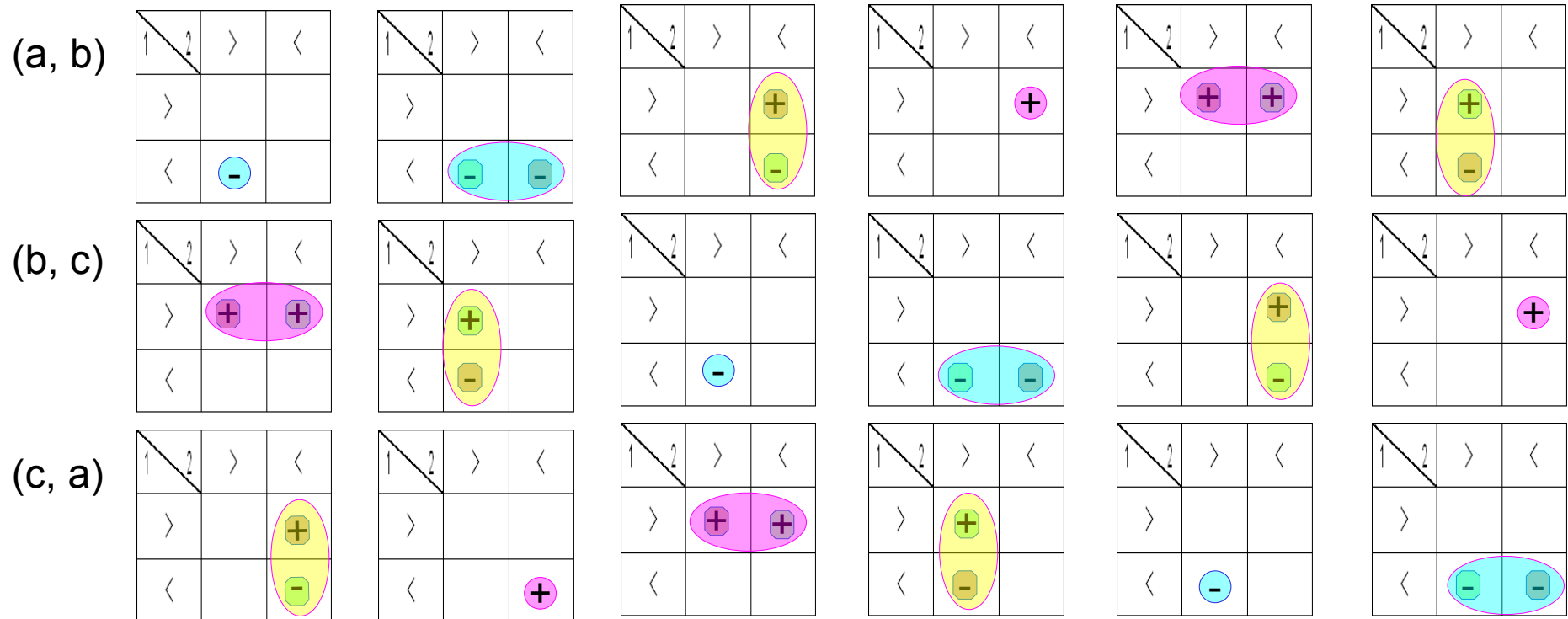
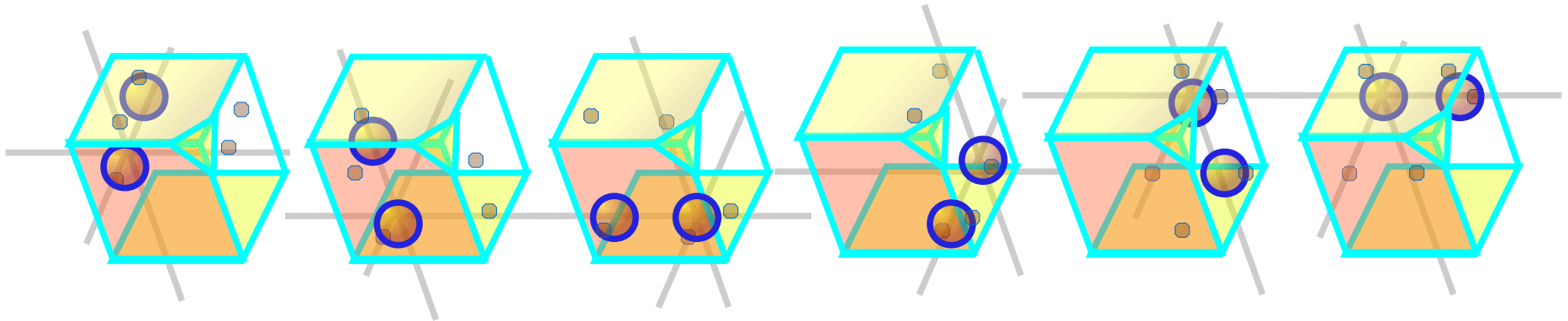
# Propagation process (1.2)







# Propagation process (2.2)



# References

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