A cube representation of social welfare function

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Aim of this slide

- Kenneth J. Arrow has developed the mathematical model of social choice and proved that dictatorship is unavoidable under a set of seemingly moderate conditions. This theorem often called the General impossibility theorem (Arrow, 1951/1963; Sen, 1994; Muolin,1989).
- Arrow's Social Welfare Function (SWF) is a function from the set of profiles of individual orderings into the set of social orderings satisfying the set of conditions, U, I, T and P, which will be explained later. This slide provides a cube representation of SWF by extending a table representation (Indo, 2007). Although the logic is essentially the same in the literature, this concise representation can provide additionally an intuitive realization of how the Arrow's theorem can be proved.

Social choice theory

- Social choice problem (eg., voting/auction/...)
  - Alternatives (ex., candidates/commodities/...)
  - Agents (ex., voters/bidders/...)
  - Agent's possible preferences (ex., complete, transitive orderings)
  - A 'profile' is a tuple of each agent's preference.
  - Social decision rule (ex,. Condorcet rule/SPA/...)

Conditions of Arrow's SWF(1)

- (T) Preference of each individual, or the society as a whole, is modeled as a linear (or weak) ordering, i.e., transitive, complete, asymmetric (or reflexive) binary relations on alternatives.
- (U) Unrestricted domain. Any profile (i.e., a combination of orderings of all agents) are possible.

Conditions of Arrow's SWF(2)

- (IIA) Independence of irrelevant alternatives. SWF is binary decomposable for each pair of alternative.
- (P) Pareto condition. Unanimity enforces the social decision.
- (ND) No-dictator. There is no unique agent who's ordering always to be a social ordering.

Arrow's theorem

- Theorem (Arrow, 1951/1963)
  - Let a model of n-agent and m-alternative, m+=3. And assume conditions U and T.
  - Then the set of conditions P, IIA, and ND for the SWF are inconsistent.
- Corollary

  P and IIA implies dictatorship (D).
- Observation

  Two dictatorial rules satisfy all these conditions.

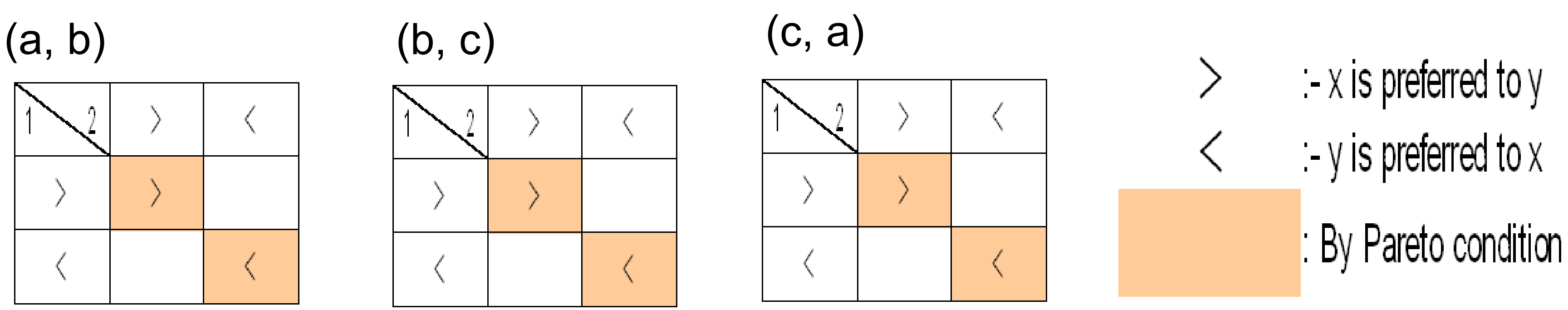
  So, D is equivalent to P and IIA assuming U and T.

Reducing into the set of orderings

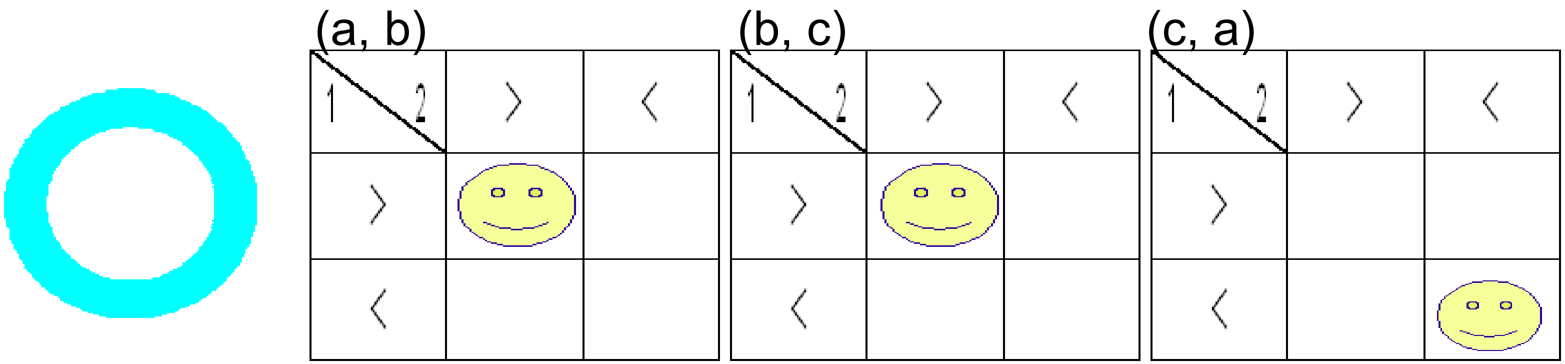
- Note that what we can learn from the Arrow's theorem is that the set of possible profiles of all individual orderings is isomorphic to the set of all possible orderings under, SWF, the aggregation rule constrained by the above conditions.
- We confine attention to a society of two individuals and three alternatives for the time being.

IIA condition and Pareto condition

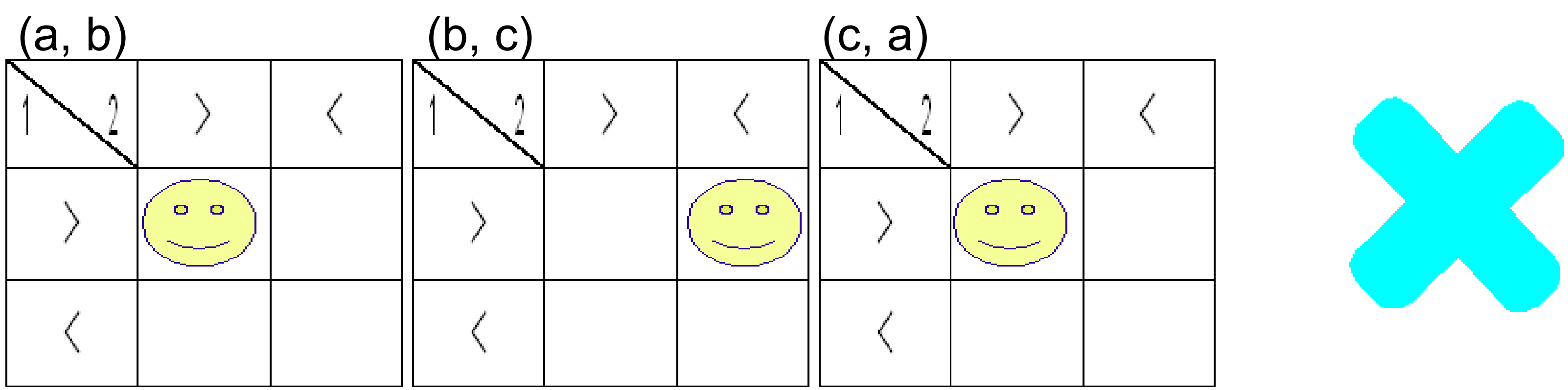
- The IIA (Independence of Irrelevant Alternative) can be seen as the background constraint to draw the propagation of the transitivity relations, which is the most important condition in Arrow's impossibility theorem.
- The social orderings (SWF) have to be decomposed into the pairwise comparisons by IIA. See three 2x2 tables in the following figure. A profile of individual orderings is to select a unique cell from each table.



Profiles and the transitivity of individual orderings

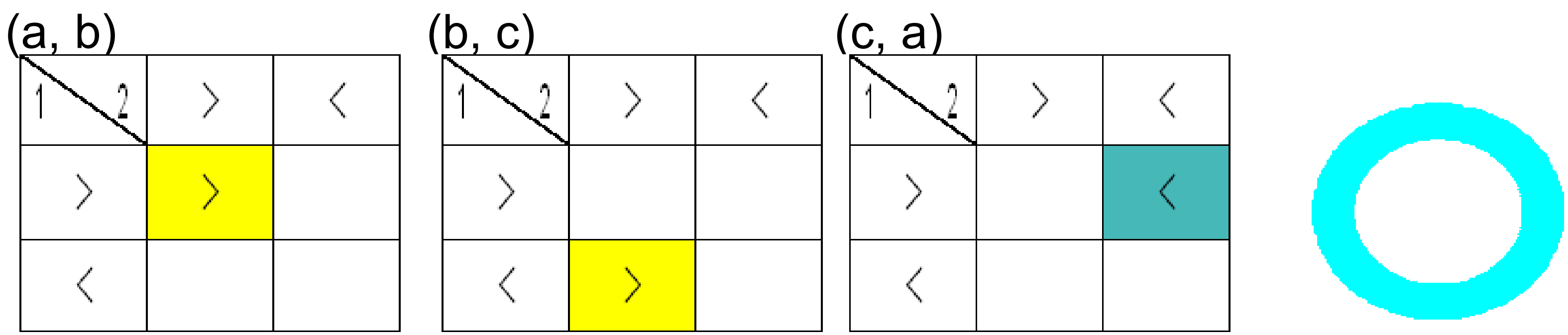


Three smiles arranged by ones for each tables represent a possible profile, 1:(a+b,b+c,c-a) and 2:(a+b,b+c,c-a), a tuple of (transitive) orderings of two agents.

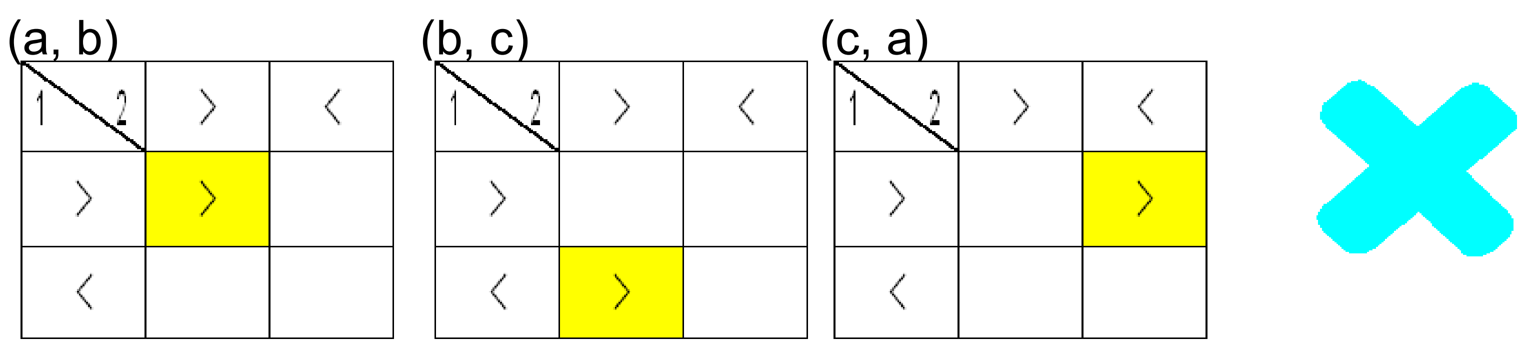


This is NOT a profile, because the ordering of row agent, 1: (a+b,b+c,c+a), is a cyclic relation, and so is intransitive.

Condition T prohibits each profile from being unilaterally directed



This can be seen as an SWF value assigned for a profile.



This can NOT be seen as a value of an SWF, because it consists a cyclic social orderings for the profile, and so is intransitive.

Condition T prohibits each profile from being unilaterally directed (2)

(a, b)

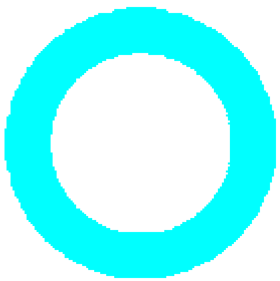
1 \ 2	>	<
>	>	
<		

(b, c)

1 \ 2	>	<
>	>	
<		

(c, a)

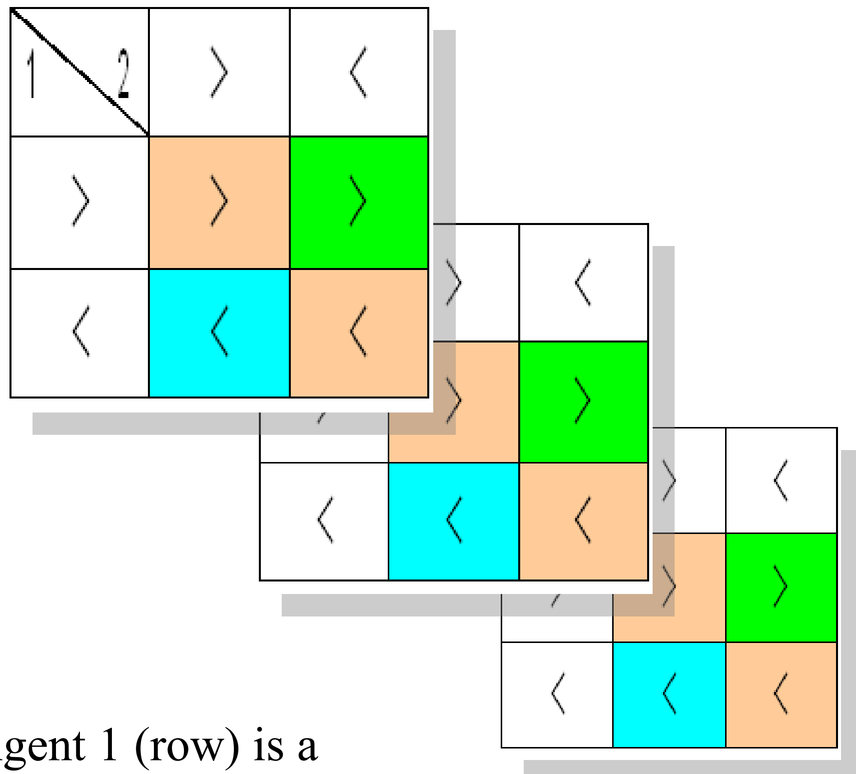
1 \ 2	>	<
>		
<	>	



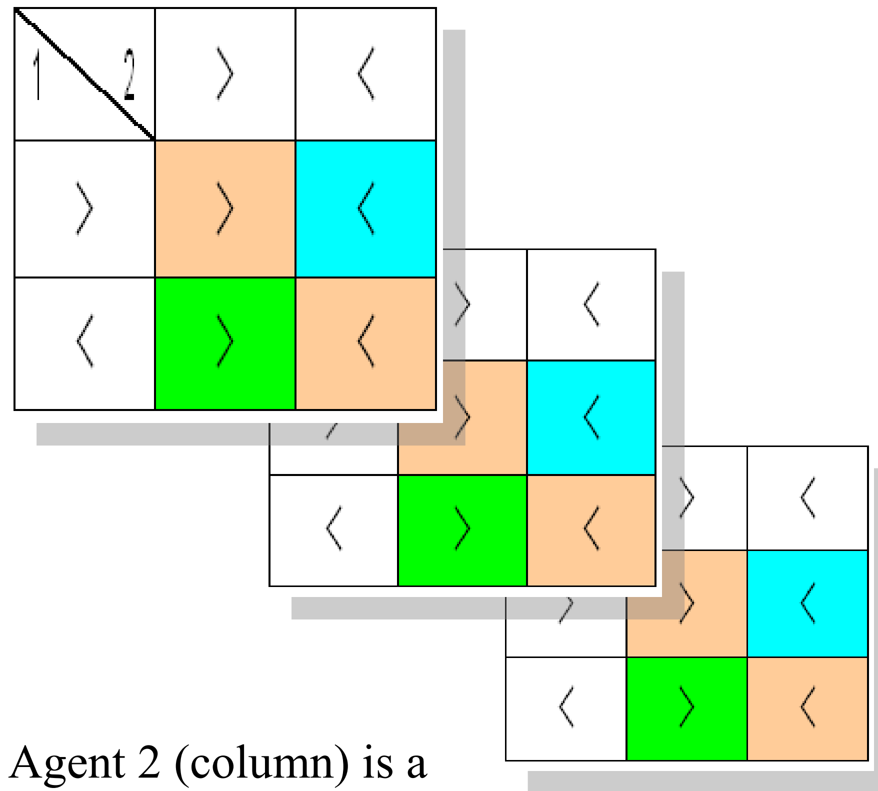
This does not violate Condition T because this is not a profile.

Two dictatorial rules

The dictatorial SWFs are clearly satisfies transitivity as well as other conditions of Arrow's theorem.



Agent 1 (row) is a dictator for this SWF.



Agent 2 (column) is a dictator for this SWF.

Conditions of SWF restated graphically

- (T & U) Individual ordering can not be selected within a single row (or a column) for each table. For each profile, which is a combination of such individual orderings, SWF should assign non-unilateral directions for each profile.
- (IIA) Profiles and SWF are represented by the three tables which are slices of the SWF with respect to directed pairs.
- (P) Diagonal elements of each table has a value which is same as the row and the column.
- (ND) There is a table which is not a simple duplications either of a row or of a column.

Proof of the theorem

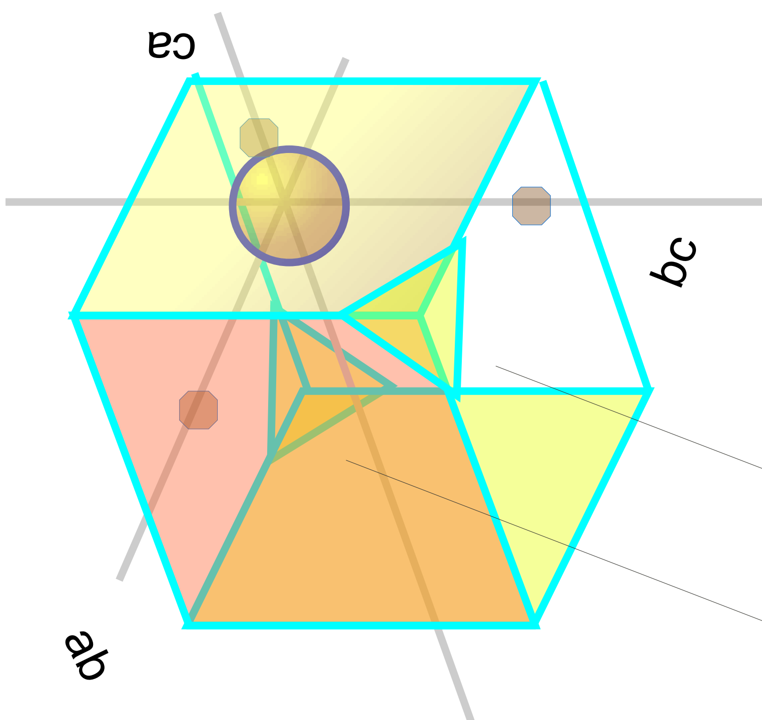
- I will present a new proof of the impossibility theorem by using a figure of a ball in a 3-dimensional cube (or equally vertexes of a cube). The conditions IIA, P, and T are concisely represented by this cube.

- And it can reduce the 36 possible profiles of linear orderings into only 6 positions where a ball is put on. They are a subset of all possible profiles, however, enough to reproduce the dictatorial result.

- This representation is novel. Although similar representation may be observed in articles written by many researchers, my graphical approach differs from them because of its pure combinatorial feature without marketplace interpretation in economics.

A cube representation of linear profile

- The following figure is one of the six cube in next slide represents an IIA-satisfied ordering profile.



(a, b)

1 \ 2	>	<
>		
<	●	

(b, c)

1 \ 2	>	<
>		●
<		

(c, a)

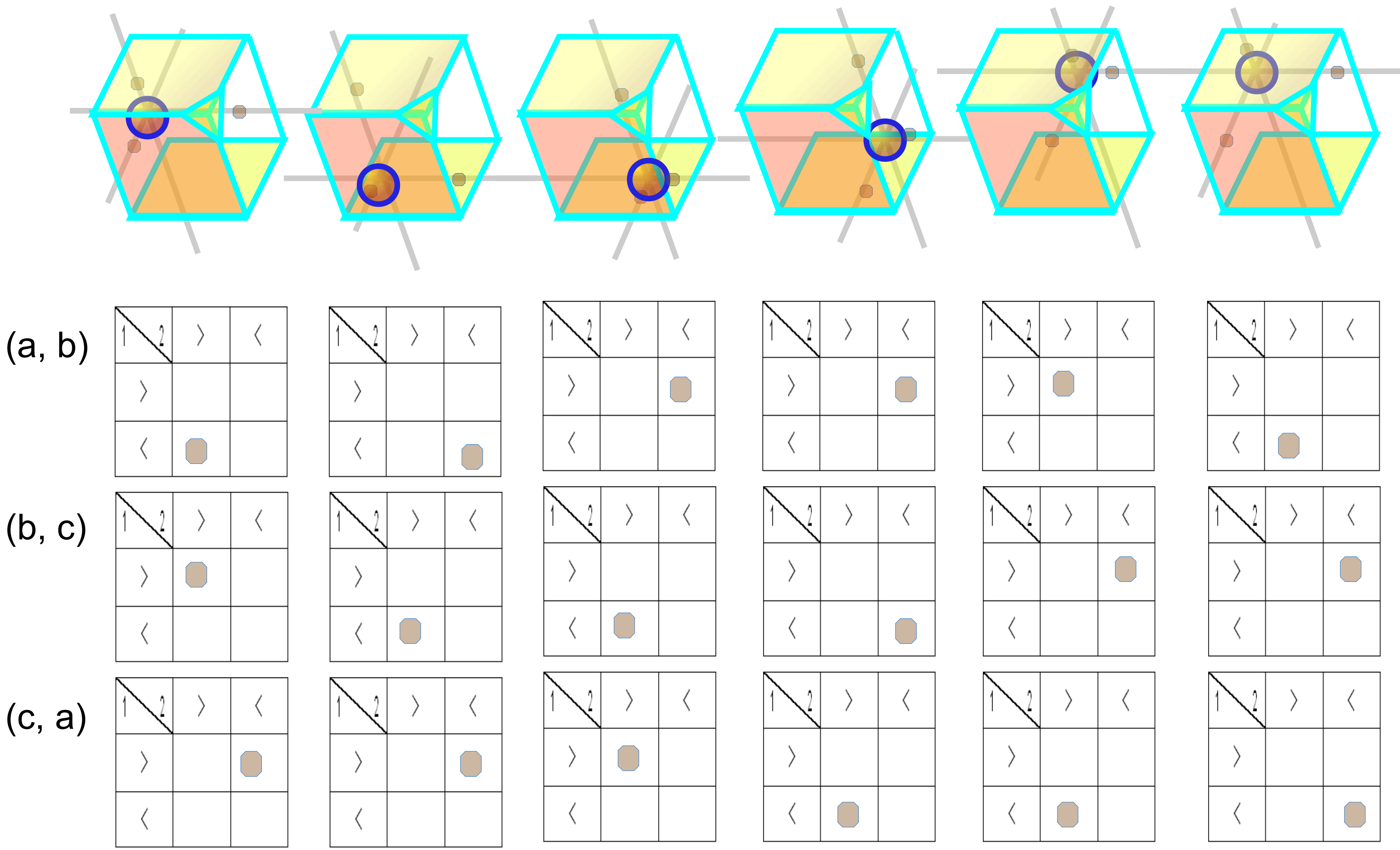
1 \ 2	>	<
>		
<		●

You may not put a ball on special two corner positions of cyclic relation.

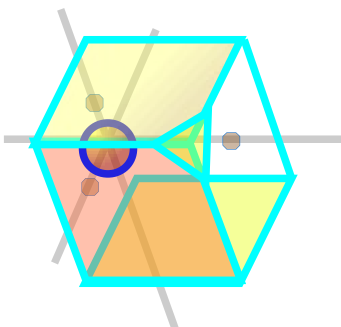
At this corner a ball would have corresponded to the All-“>”-position

Another diagonal corner shows the All-“<”-position

Ordering profiles of six cubes with a ball



Transitivity propagation schema

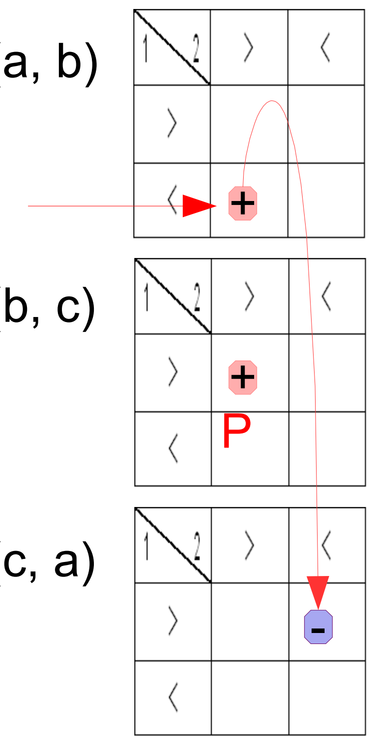


- Transitivity (Condition T) inhibits all-same-vale, i.e., a cycle, for each profile. Therefore, if the value for a pair (a, b) and the binary profile (<, >) is “+”, which represents “>”-social relation, in the figure left, then the value for a pair (c, a) at (>, <) can not be a “+” without a violation to Condition T.

(a, b)

(b, c)

(c, a)

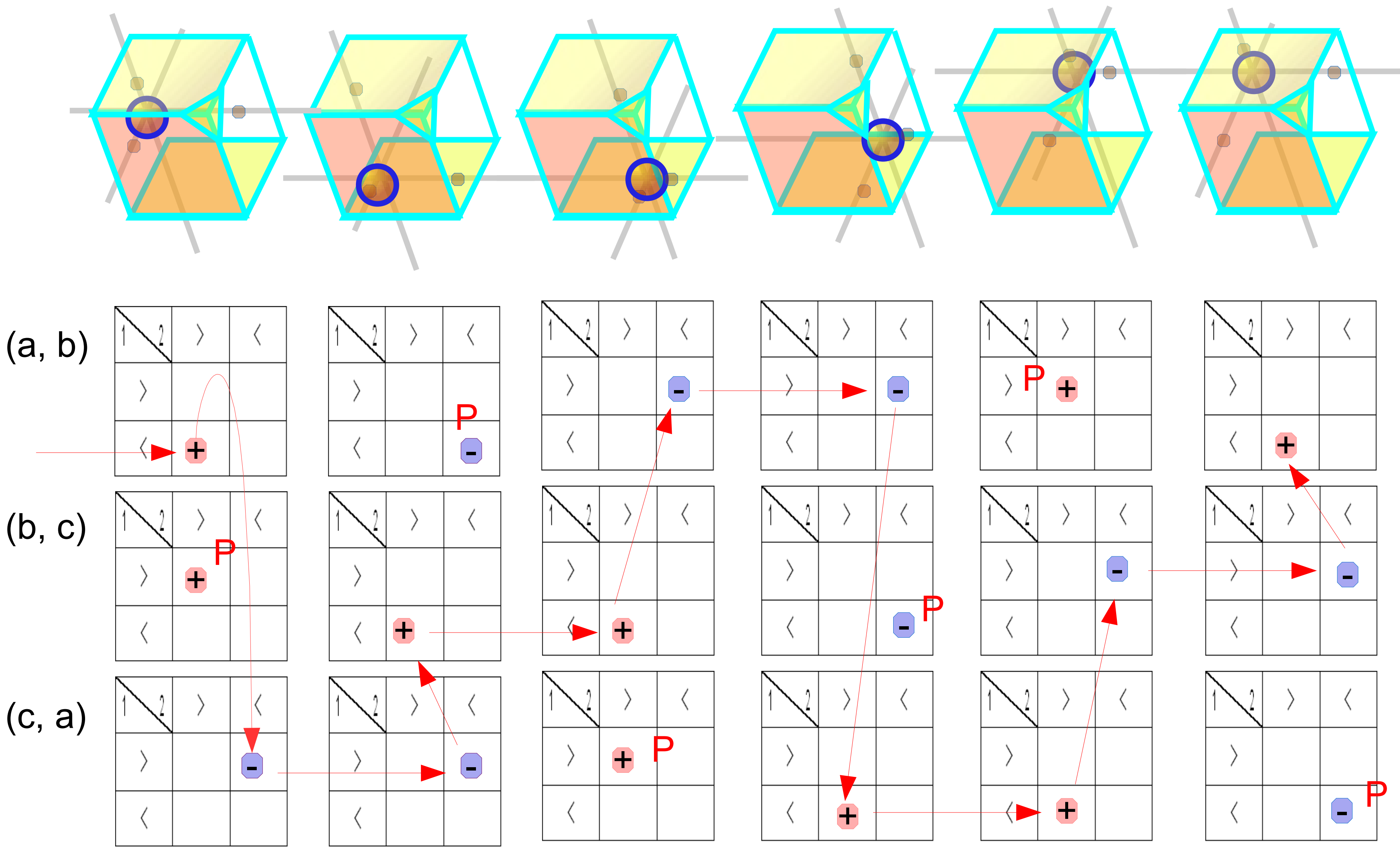


+ :”>”-relation

- :”<”-relation

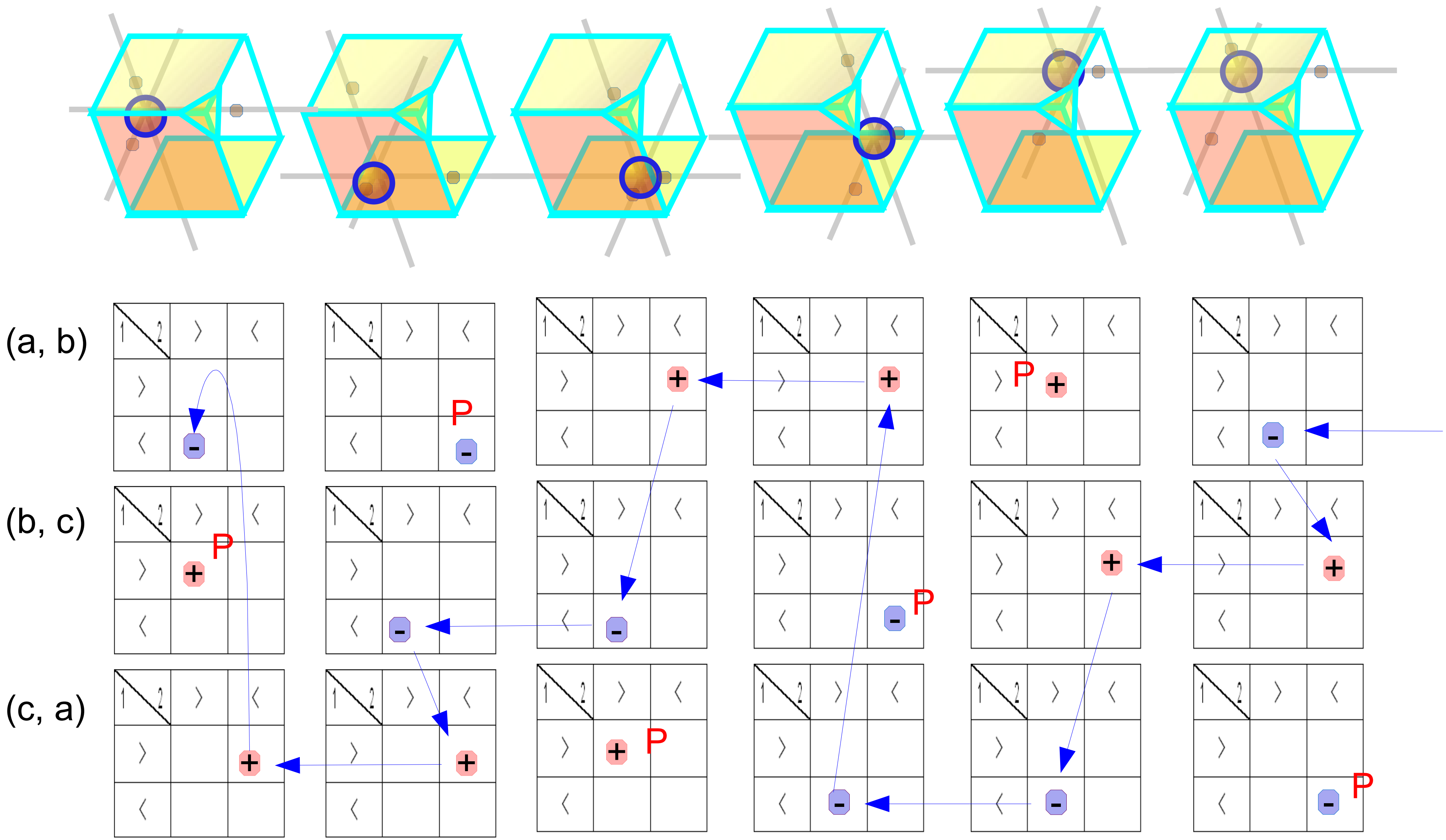
P: Predetermined by Pareto condition

Propagating transitive relations among cubes (1)



P: Predetermined by Pareto condition   + :">"-relation   - :"<"-relation

Propagating transitive relations among cubes (2)



P: Predetermined by Pareto condition   + :">"-relation   - :"<"-relation

Decisive set

- As shown in the literature (Arrow, 1963;Sen, 1995) the proof can be simplified by a notion of decisive set (or ultra-filter) and its decomposable nature. It is said that a subgroup D is decisive for a directed pair (x, y) if the subgroup unanimously prefer x to y then the society should prefer x to y. Note that the propagation schema in the preceding slides shows that a decisiveness for a single pair prevails via a profile such that $(x>y>z)$& $(z>x>y)$. The same reasoning in the preceding slides can be applied to more than three alternatives by renaming the symbols. By Condition IIA, it suffices to prove that for any two triples, for (1,2,3) and (2,3,4), which share a pair, (2,3), there can not be a different (local) dictator, a singleton decisive set.

proof by using cube

- If we restrict our attention to the set of subgroup-unanimous profiles, the figure of cubes comprising three tables and a ball are analogously same as before. Decisive subgroup D corresponds to the row and another subgroup C=N-D corresponds to the column respectively. The SWF-value at a ball is constrained as before. Namely, a dictatorial rule of the row. However, subgroup C is a set of dummy players, we can remove it from N and create a new cube by dividing D into a pair of two subgroups, A and B, D=AUB, B=C-A. By applying the above subgroup-unanimous-cube argument again, we can see that either A or B would be decisive. Recursive application of this argument leads to a singleton set.

References

- K. J. Arrow (1951/1963). *Social Choice and Individual Values*, Yale University Press.
- H. Moulin (1988). *Axioms of Cooperative Decision Making*, Cambridge University Press.
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