A graphical representation of Arrow's theorem

Kenryo Indo

Kanto Gakuen University

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Aim of this slide

- Arrow's Social Welfare Function (SWF)
    - a function from the set of profiles of individual orderings into the set of social orderings satisfying the set of conditions which will be explained later.
- general impossibility theorem (Arrow, 1963).
    - Kenneth J. Arrow has developed the mathematical model of social choice and proved that dictatorship is unavoidable under a set of seemingly moderate conditions (i.e., the general impossibility theorem).
    - In this slide I will provide a graphical proof for the impossibility theorem under linear orderings for 2-agent and 3-alternative cases. It can be intuitively understood, however, without loss of rigor.

Social choice theory

- Social choice problem (eg., voting/auction/...)
  - Alternatives (ex., candidates/commodities/...)
  - Agents (ex., voters/bidders/...)
  - Agent's possible preferences (ex., complete, transitive orderings)
  - A 'profile' is a tuple of each agent's preference.
  - Social decision rule (ex,. Condorcet rule/SPA/...)

Conditions of Arrow's SWF

- (T) Preference of each individual, or the society as a whole, is modeled as a linear (or weak) ordering, i.e., transitive, complete, asymmetric (or reflexive) binary relations on alternatives.
- (U) Unrestricted domain. Any profile (i.e., a combination of orderings of all agents) are possible.
- (IIA), (P), (ND) => next slide

Conditions of Arrow's SWF(2)

- (T), (U) => preceding slide
- (IIA) Independence of irrelevant alternatives. SWF is binary decomposable for each pair of alternative.
- (P) Pareto condition. Unanimity enforces the social decision.
- (ND) No-dictator. There is no unique agent who's ordering always to be a social ordering.

Arrow's theorem

- Theorem (Arrow, 1951/1963)
  - Let a model of n-agent and m-alternative, m>=3. And assume conditions U and T.
  - Then the set of conditions P, IIA, and ND for the SWF are inconsistent.
- Corollary

  P and IIA implies dictatorship (D).
- Observation

  Two dictatorial rules satisfy all these conditions.

  So, D is equivalent to P and IIA assuming U and T.

Binary decomposition which naturally represents the IIA condition

(a, b)

1 \ 2	>	<
>		
<		

(b, c)

1 \ 2	>	<
>		
<		

(c, a)

1 \ 2	>	<
>		
<		

For each pair (x, y),

$>$:- x is preferred to y

$<$:- y is preferred to x

The weak Pareto condition (unanimity)

(a, b)

1 \ 2	>	<
>	>	
<		<

(b, c)

1 \ 2	>	<
>	>	
<		<

(c, a)

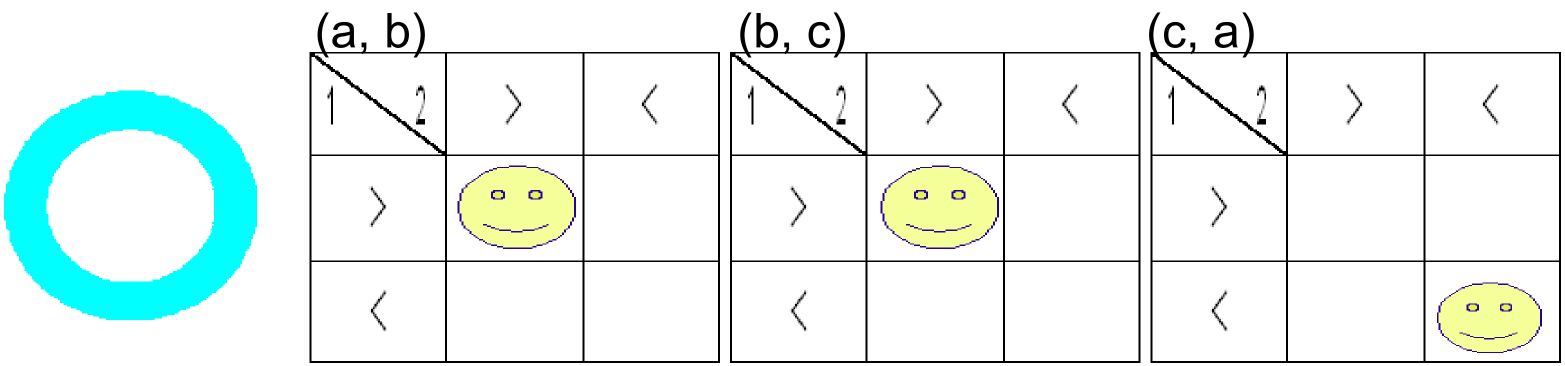
1 \ 2	>	<
>	>	
<		<

> :- x is preferred to y

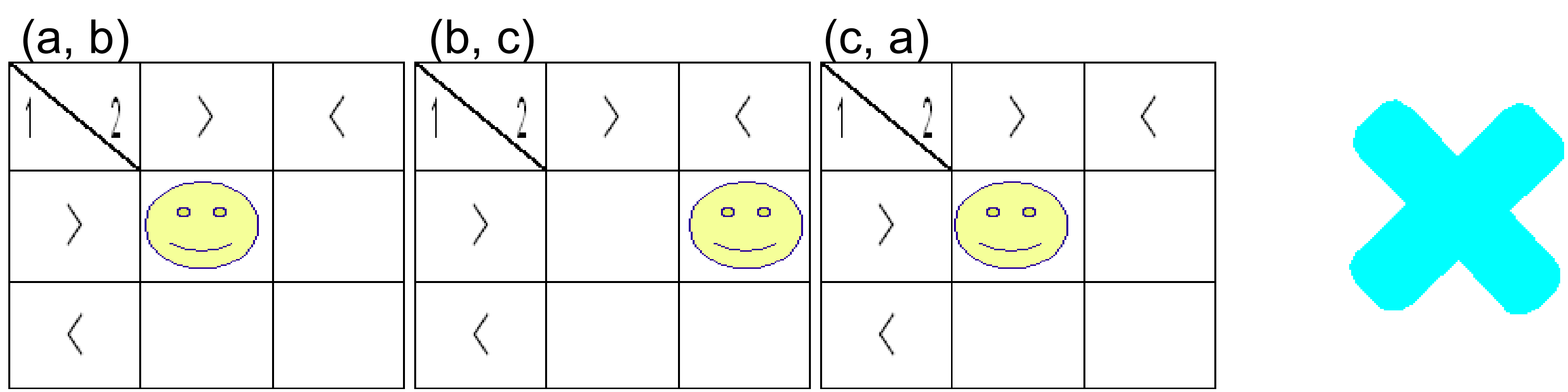
< :- y is preferred to x

(shaded) : By Pareto condition

Profiles and the transitivity of individual orderings



Three smiles arranged by ones for each tables represent a possible profile, 1: (a>b,b>c,c<a) and 2:(a>b,b>c,c<a), a tuple of (transitive) orderings of two agents.



This is NOT a profile, because the ordering of row agent, 1: (a>b,b>c,c>a), is a cyclic relation, and so is intransitive.

Condition T prohibits each profile from being unilaterally directed

(a, b)

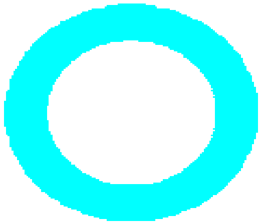
1 \ 2	>	<
>	>	
<		

(b, c)

1 \ 2	>	<
>		
<	>	

(c, a)

1 \ 2	>	<
>		<
<		



This can be seen as an SWF value assigned for a profile.

(a, b)

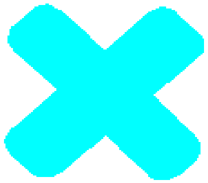
1 \ 2	>	<
>	>	
<		

(b, c)

1 \ 2	>	<
>		
<	>	

(c, a)

1 \ 2	>	<
>		>
<		



This can NOT be seen as a value of an SWF, because it consists a cyclic social orderings for the profile, and so is intransitive.

Condition T prohibits each profile from being unilaterally directed (2)

(a, b)

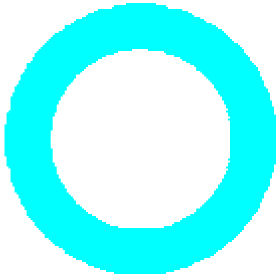
1＼2	>	<
>	>	
<		

(b, c)

1＼2	>	<
>	>	
<		

(c, a)

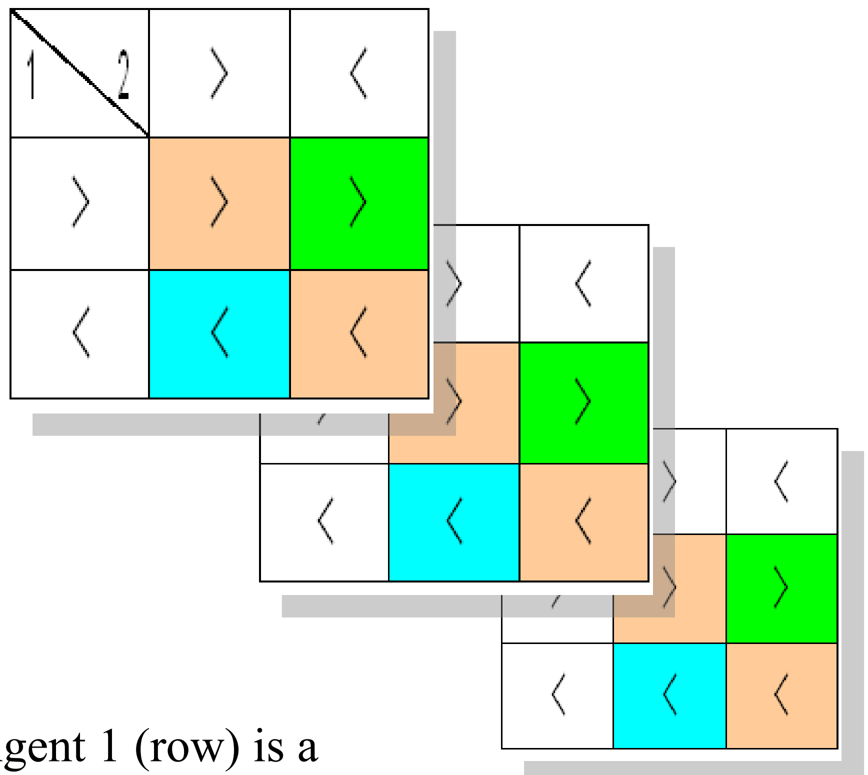
1＼2	>	<
>		
<	>	



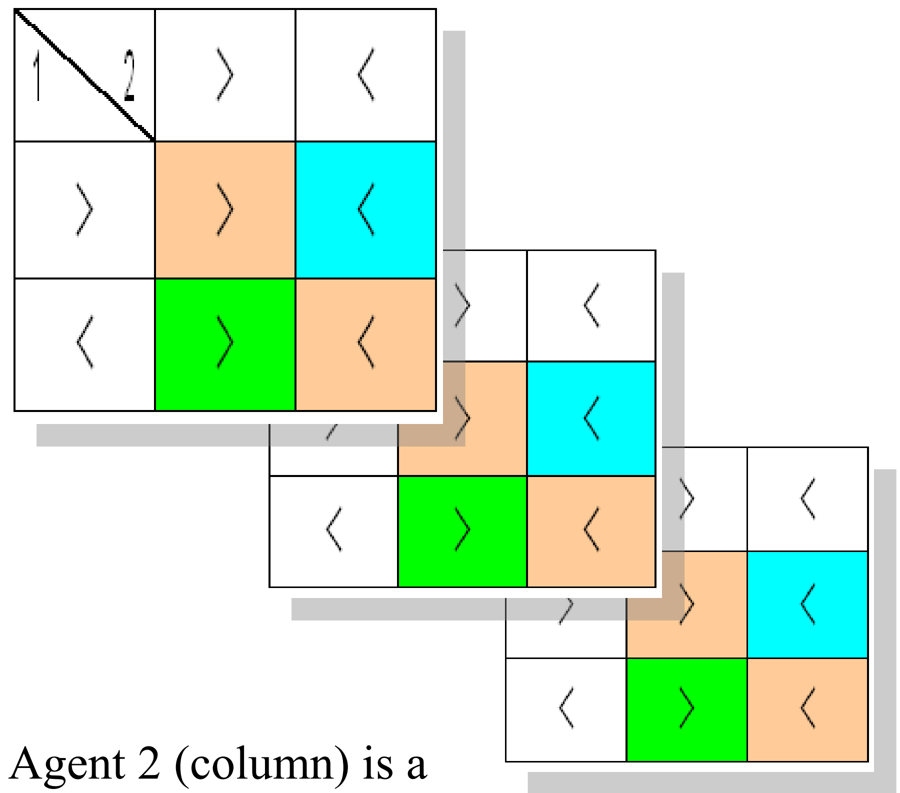
This does not violate Condition T because this is not a profile.

Two dictatorial rules

The dictatorial SWFs are clearly satisfies transitivity as well as other conditions of Arrow's theorem.



Agent 1 (row) is a dictator for this SWF.

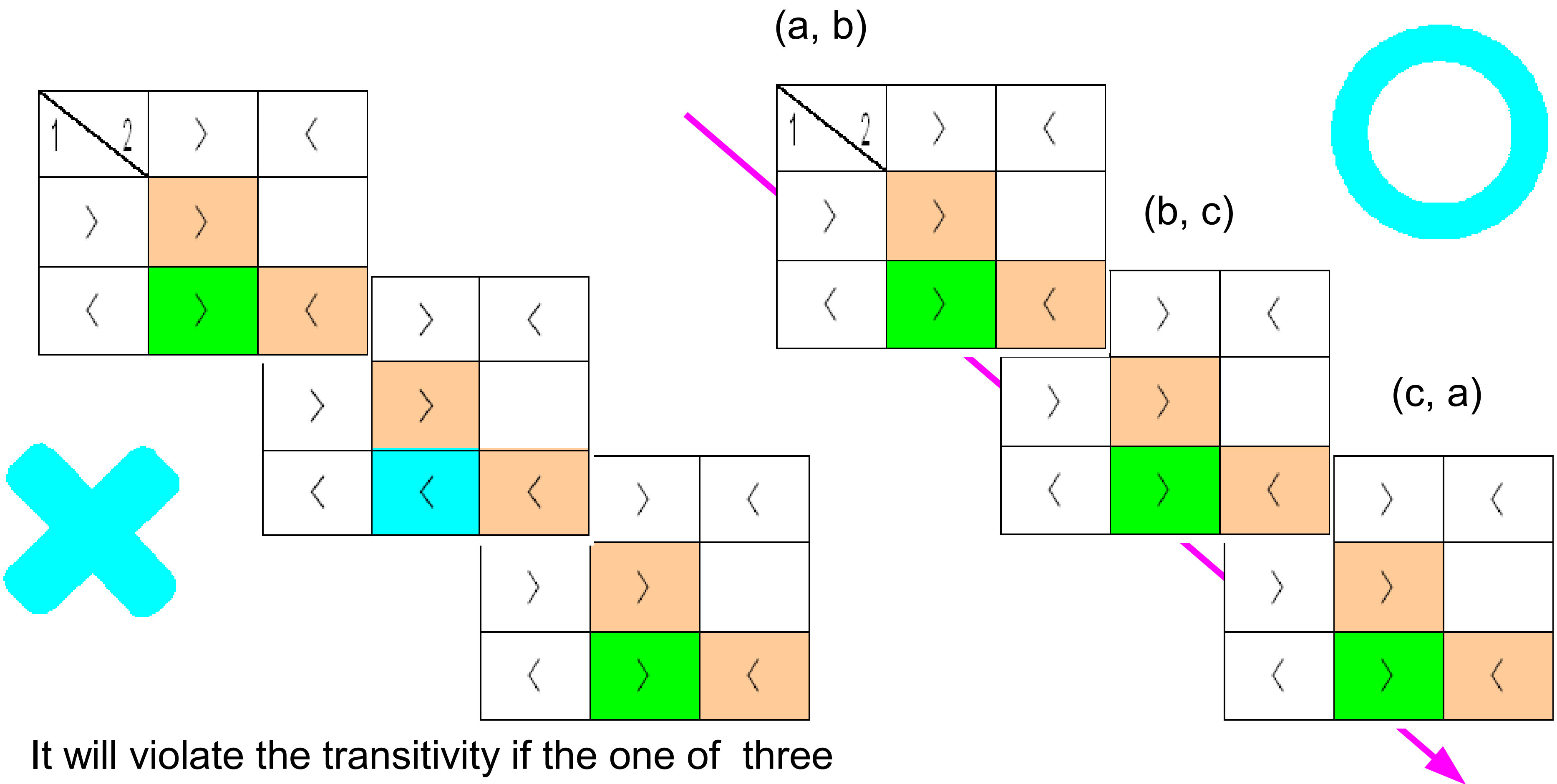


Agent 2 (column) is a dictator for this SWF.

Conditions of SWF restated graphically

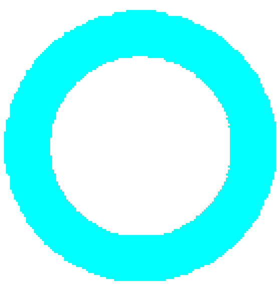
- (T & U) Individual ordering can not be selected within a single row (or a column) for each table. For each profile, which is a combination of such individual orderings, SWF should assign non-unilateral directions for each profile.
- (IIA) Profiles and SWF are represented by the three tables which are slices of the SWF with respect to directed pairs.
- (P) Diagonal elements of each table has a value which is same as the row and the column.
- (ND) There is a table which is not a simple duplications either of a row or of a column.

Condition T requires all tables to have a same single direction pushing through each non-diagonal cell (lemma 1)



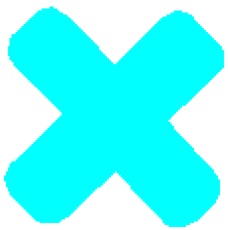
It will violate the transitivity if the one of three tables replaced with the above one.

Condition T implies that different non-diagonal elements should not be unilateral for each table (lemma 2)



1 \ 2	>	<
>	>	>
<	<	<

1 \ 2	>	<
>	>	<
<	<	<



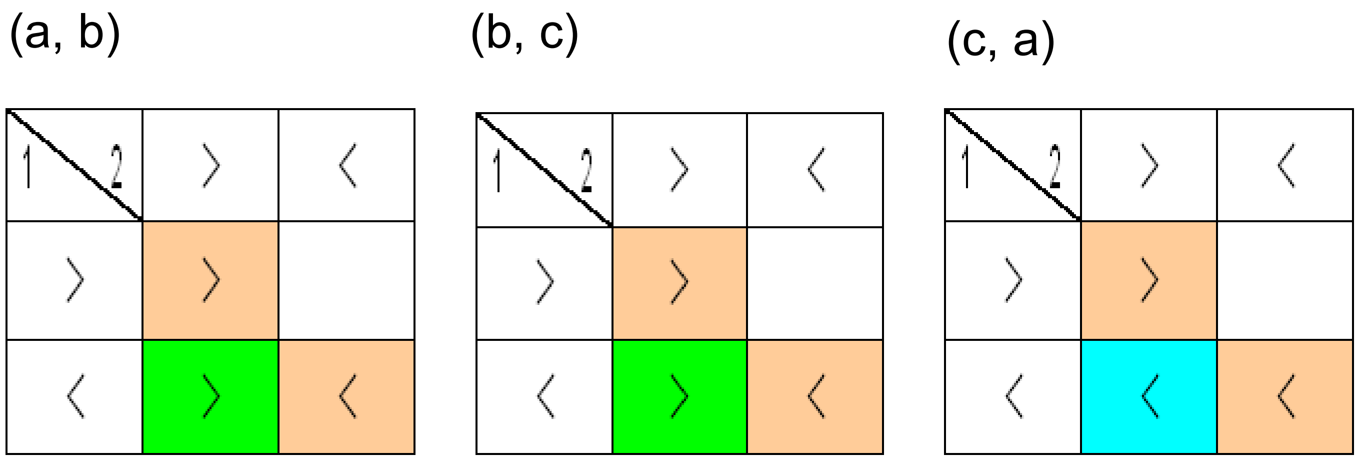
You can not burn the candle at the both ends. It can be proved that it violates the transitivity!

Proof of the theorem

- Dictatorial rules clearly satisfies the conditions of SWF and above two lemmas.
- Obviously, lemma 1 and lemma 2 together complete a proof of the dictatorial result (and so of the impossibility theorem).

Proof (lemma 1)

I insist that we can suppose the following pattern of the SWF without loss of the generality. Then, I will prove that it goes to violate the transitivity.



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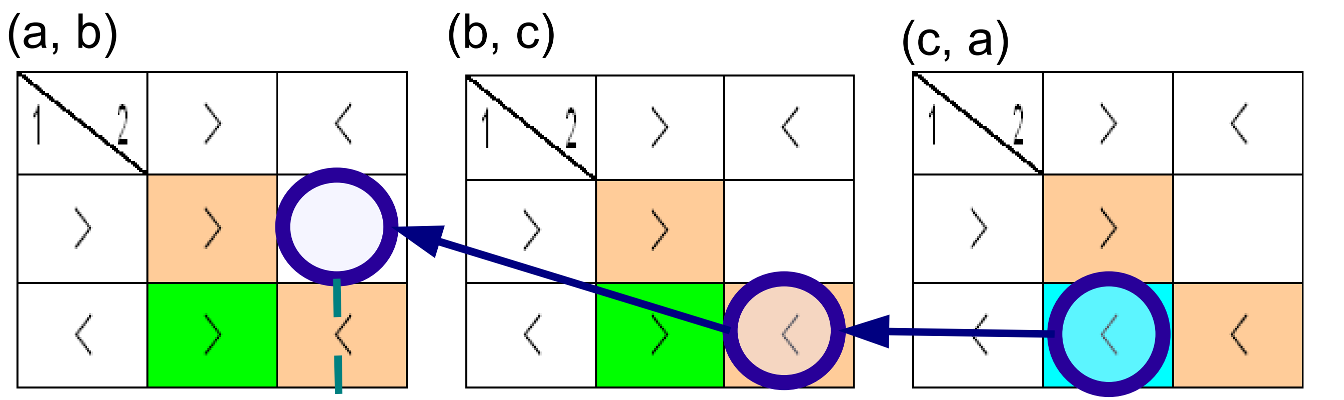
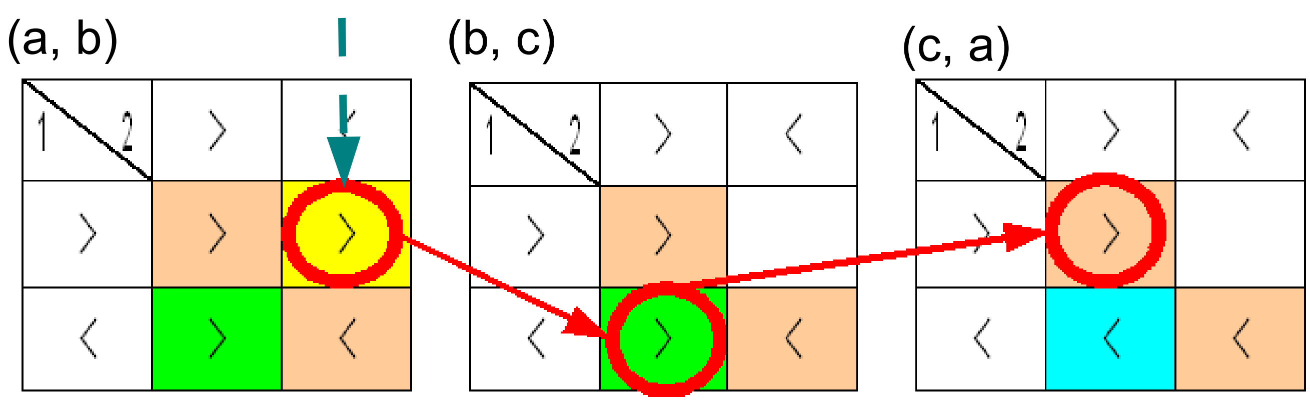
Proof (lemma 1) continued

Let us pick up a profile ((>,<,<), (<,<,>)). Then the value of the SWF must be a>b in order to satisfy Condition T.

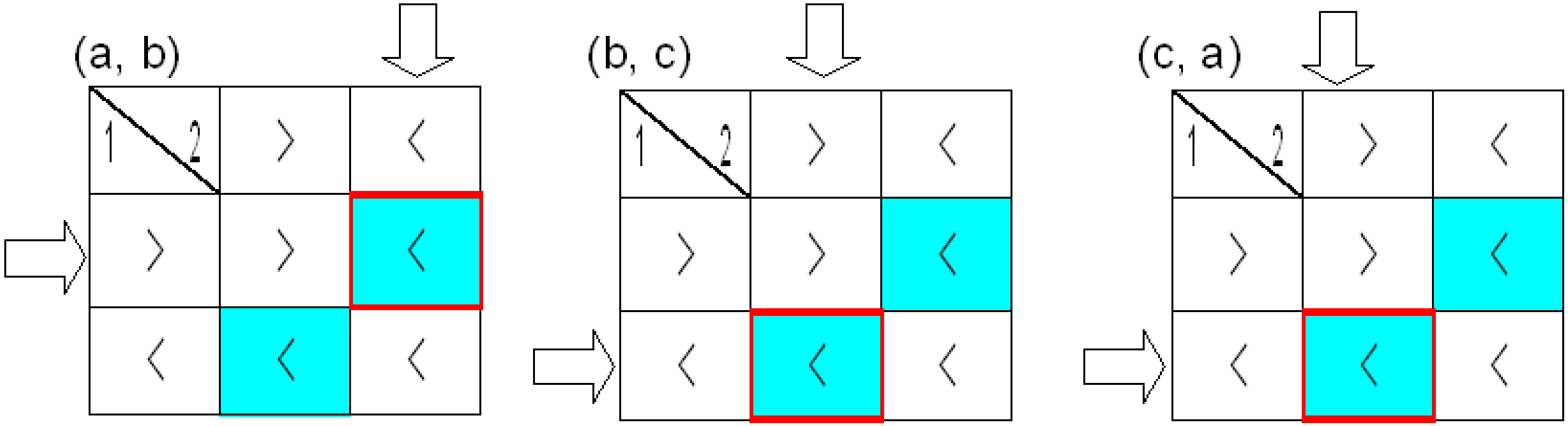


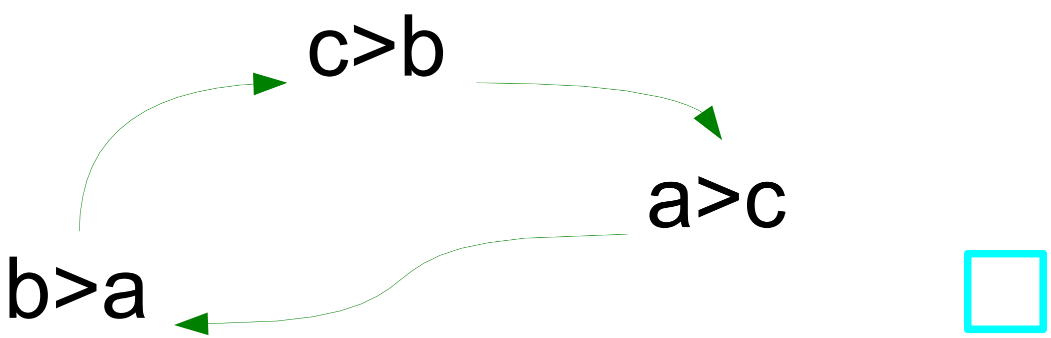
However it contradicts Condition T because a profile can be selected as shown in the following figure which shows an intransitive social ordering.



Proof (lemma 2)

Suppose a profile (a>c>b, b>a>c). By lemma 1, it suffices to consider an SWF like as the following pattern. This pattern results in a cyclic relation, so it can not be a social ordering.





References

- K. J. Arrow (1951/1963). Social Choice and Individual Values, Yale University Press.
- A. Sen (1995). Rationality and social choice. American Economic Review 85(1):1-24.