A graphical representation of Gibbard-Stterthwaite theorem

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Aim of this slide

- This slide gives a graphical proof for the Gibbard-Satterthwaite theorem for 2-agent and 3-alternative social choice model.
- And I would like to press the point, as a supplement, a relation to Nash implementation theory regarding Maskin-monotonicity.

Social Choice Function (SCF) and problem of manipulation

- SCF is a function from the set of profiles of individual orderings (and the set of subsets of states) into the set of states of the society.
- A SCF f is (strategically) manipulable if there is a profile and an individual who can be benefited from disguising a false preference.

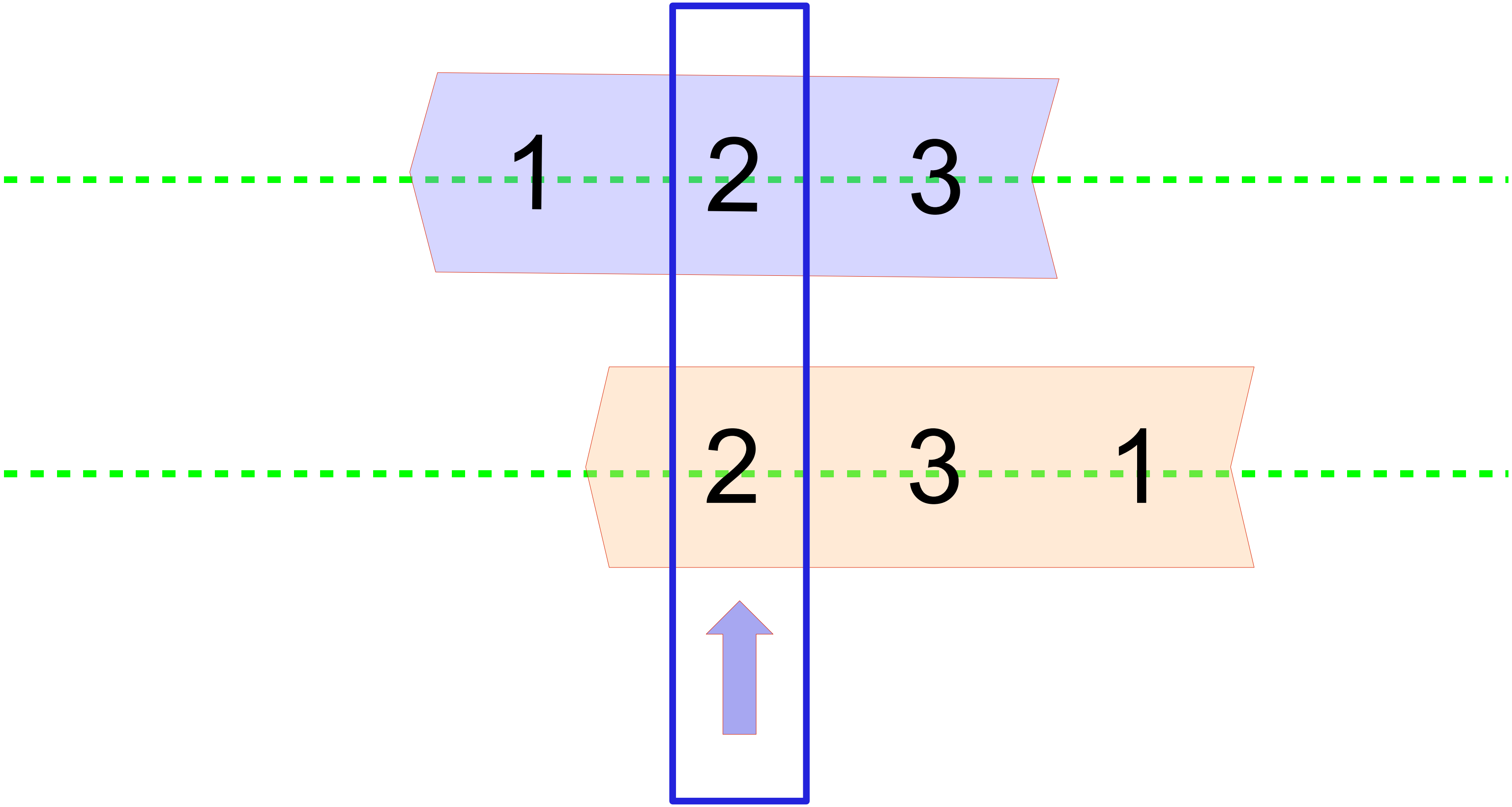
Gibbard-Satterthwaite theorem

- A famous theorem proved independently by A. Gibbard and M. Satterthwaite in 1970s, which is closely related to the general impossibility theorem, dictatorship is unavoidable if non-manipulable (i.e., strategy-proof).

Model of social choice

- X: a set of alternatives, or possible states the group N (or it's planner) can select. (ex., candidates/commodities/...)
- N: the entire group of individuals who have common concern about the choice from X.(ex., voters/bidders/...)
- R: society and individual's possible preferences (ex., complete, transitive orderings). A 'profile' is a tuple of each agent's preference.
- Social Choice Function (SCF)
  - A function, $f:R^n \text{->} X$, which maps each possible profile of individual orderings to an alternative.

SCF : Graphical view at a profile



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A dictatorial SCF in tabular form

R \ C	123	132	213	231	312	321
123	1	1	1	1	1	1
132	1	1	1	1	1	1
213	2	2	2	2	2	2
231	2	2	2	2	2	2
312	3	3	3	3	3	3
321	3	3	3	3	3	3

- Numbers, 1, 2, and 3, in above figure represent the states of the society. And three digit numbers in the labels of rows and columns respectively represent the possible preferences of the two individuals. We will abuse R and C as the ordering of each individual respectively.

Conditions for SCF

- (T) Assume the preferences of every individuals are transitive, complete, asymmetric binary relations (liner orderings or rankings without tie) on the alternatives.
- (U) Unrestricted domain. Any combination, i.e., a profile, of orderings of all individuals are possible.
- (SP) There are no profiles and individuals can manipulate their own true preference (in the direct revelation game).

Conditions for SCF(2)

- (Z) Citizen sovereignty, or no taboo alternative. For each state, there is at least one profile at which the state is selected by the SCF.
- (P) Unanimity decides the SCF outcome.
- (D) Dictatorship. There is a single unique individual, a dictator, who's best alternative is selected by the SCF for each profile.
- (ND) There is no dictator.

the theorem

- Theorem (Gibbard, 1973; Satterthwaite, 1975)
  - Let a model of n-agent, m-alternative, where n>=2 and m>=3.
  - Assume conditions U, T, and Z.
  - Then, SP is not compatible with ND. Any SCF which satisfies SP is dictatorial (D).

Dictatorial SCF

- Observation

  Two dictatorial rules trivially satisfy these conditions required in the theorem.

Decompose profiles into the blocks of maximal elements

R \ C	123 132	213 231	312 321
123 132	*A*	*B*	*C*
213 231	*D*	*E*	*F*
312 321	*G*	*H*	*I*

Lemma 1. the Pareto condition satisfied and each diagonal block is single-valued

R \ C	123 132	213 231	312 321
123 132	*1*	*B*	*C*
213 231	*D*	*2*	*F*
312 321	*G*	*H*	*3*

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Proof of Lemma 1: step 1

- It satisfies SP at least within a block if single-valued. If x=1 then it must be the case (i.e., x=y=z=w=1), otherwise clearly manipulable.
- Suppose x≠1. Then neither of y, z, and w to be 1. If x=2, y=z=3 or a single-2-valued can be considered, but the former should be excluded, for if w= 2 or 3 then manipulable. If x=3 then only a single-3-valued is allowed.

R \ C	123	132
123	*x*	*y*
132	*z*	*w*

For each remaining diagonal blocks, it will be proved by permutation of numbers.

Proof of Lemma 1: step 2

- By previous argument, three diagonal blocks should be all single-valued. Next I prove the such blocks consist of each agreed top values. And it would suffice to prove a case of Block A.
- Suppose $x \neq 1$ for Block A. Then there is no 1-value in any of blocks A, B, C, D, and G. By Condition Z, at least one of the remaining four blocks E, F, H, or I includes it. But it makes manipulable to get the 1-value at some profile either in A, B, C, D, or G.

Lemma 2. Each row block (and column block) should be either single-valued or all-different

R \ C	123 132	213 231	312 321
123 132	A	A	A
213 231			
312 321			

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R \ C	123 132	213 231	312 321
123 132	A		
213 231	B		
312 321	C		

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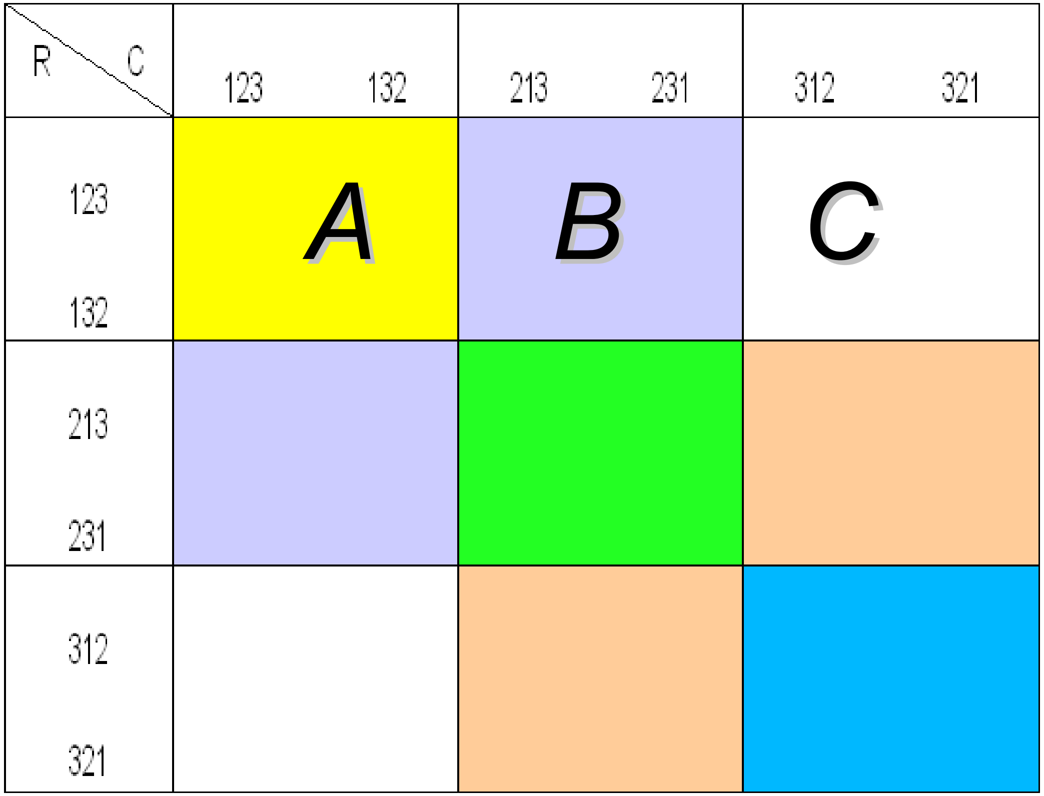
Proof of Lemma 2: outline

- It is sufficient to prove for a row of three blocks, A, B, and C in the following figure. Remaining rows and columns can be proved by permutation of numbers.



Proof of Lemma 2: step 1

- For A, it has been proved by Lemma 1. Note that except for single-valued, it does not violate Condition SP only if (x, z)=(y, w)=(2,3) for both B and C. Non-constancy requires (x, y)=(z, w)=(1, 3) for B, or (1, 2) for C, which is manipulable respectively (See figure below).

B:

R \ C	213	231
123	1	3
132	1	3

C:

R \ C	312	321
123	2	2
132	3	3

Proof of Lemma 2: step 2

- By the constancy of Block A, it immediately follows that we can not assign (x, z)=(y, w)=(2,3) for blocks B and C. There are two and only two possibilities. First, the three blocks are all single-valued of 1. Second, A, B and C are single 1-2-3 valued respectively.

R \ C	123	132	213	231	312	321
123	1	1	2	2	2	2
132	1	1	3	3	3	3

Lemma 3. If a row (or column) block is single-valued or all-different then the SCF is dictatorial

R \ C	123 132	213 231	312 321
123 132	1	1	1
213 231	2	2	2
312 321	3	3	3

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Proof of Lemma 3

- Trivial. □

Note that a pattern shown as right figure, which satisfies Lemma 2 without Lemma 1, of cause, is not strategy-proof.

R \ C	123 132	213 231	312 321
123 132	1	2	3
213 231	2	3	1
312 321	3	1	2

Proof of the GS theorem

- It can be easily proved according to the series of lemma in previous slides.

Supplement: Condition M

- (M) Maskin-Monotonicity. For each state, x, chosen by the SCF at a profile, p, x holds as the value of SCF for any other profile, q, unless there is an individual and another state, y, the individual's preference reversed.

M iff SP

- Let p and q are two profiles such that, q=p/q(i), q can be derived from p by changing agent i's preference. Suppose the SCF is manipulable, SCF(p)=x, SCF(q)=y, and i prefers y to x at p.
- So the proof of M-->SP is trivial, however the converse is not.

$\neg M \implies \neg SP$

- Note that a violation to the monotonicity means that there is a profile at which each individual can find a Pareto improvable false profile. Then it can be shown that there is at least for one individual who is motivated to fraud nevertheless it may harm the opponent's benefit (See Muller and Satterthwaite, 1977).

Nash implementation

- As we have already seen, if the SCF violates Condition M, the true profile can not be a Nash equilibrium not only in direct revelation, but also jointly revelation (profile designation).
- Condition M turned out be necessary to Nash implementation and a sort of joint revelation with auxiliary messages may suffice, as discovered by Eric S. Maskin and has been elaborated by many other researchers (See Maskin(1999)).

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